CALCULUS ON MANIFOLDS

1. Exam

Problem 1. A tuple of smooth vector fields $X_1, \dots, X_k$ on a smooth n-dimensional manifold M is *involutive*, if they are linear independent at each point $a \in M$ and the Lie bracket of any two vector fields belongs to the linear span of these fields, $[X_i, X_j] = \sum_k \alpha_{ijk} X_k$ for some functions $\alpha_{ijk} \in \mathscr{C}(M)$.

1. Prove that any other tuple $Y_1, \dots, Y_k$ obtained from X_i by a nondegenerate linear transformation, $Y_i = \sum A_{ij} X_j$, $A_{ij} \in \mathscr{C}(M)$, with $\det \|A_{ij}(x)\|$ nowhere vanishing, is also involutive.
2. Give examples of involutive tuples.

A tuple of vector fields $X_1, \dots, X_k$ is *integrable*, if for any point $a \in M$ there exists a piece of k-dimensional smooth submanifold N passing through a, such that all vector fields X_i are tangent to N. These submanifolds are called *integral submanifolds*.

3. Give examples of integrable tuples.
4. Prove that each integrable tuple is involutive.
5. Prove that a tuple which consists of a single nonvanishing vector field, is both integrable and involutive.
6. Assume that the tuple consists of *commuting* vector fields, $[X_i, X_j] = 0$ for all $i, j = 1, \dots, k$. Prove that it is involutive by explicitly constructing the integrable submanifolds using the flows of these fields.
7. Let $X_1, \dots, X_k$ be an involutive tuple in $\mathbb{R}^n$ locally near the origin. Show that there exists a system of local coordinates split into the two groups, "horizontal" $(u_1, \dots, u_k)$ and "vertical" $(v_{k+1}, \dots, v_n)$, and a tuple of linear independent vector fields $Y_1, \dots, Y_k$, such that $Y_i = \frac{\partial}{\partial u_i} + V_i$, where V_i are tangent to the "vertical" direction and the tuples $\{X_i\}$ and $\{Y_i\}$ span the same subspace as in the item 1.
8. Prove that the tuple $\{Y_i\}$ is commutative as in item 5.
9. Prove that any involutive tuple of vector fields is integrable.

Problem 2. A differentiable 1-form $\omega \in \Omega^1(M)$ is called involutive, if $\mathrm{d}\omega$ is divisible by ω, that is, there exists a 1-form $\eta \in \Omega^1(M)$ such that $\mathrm{d}\omega = \omega \wedge \eta$.

1. Prove that an involutive form is integrable, that is, for any point $a \in M$ exists…(complete the statement).

2. Can you guess what should be the definition of involutivity for a tuple of 1-forms such that involutivity implies the integrability? Use the word "ideal".

Problem 3. The real 2-torus $\mathbb{T}^2$ is defined as the quotient space $\mathbb{R}^2/\mathbb{Z}^2$, hence the space $\mathscr{C}(\mathbb{T}^2)$ can be identified with the space of double periodic smooth functions on $\mathbb{R}^2$.

1. From this description calculate the first and second de Rham cohomology of $\mathbb{T}^2$.

The Möbius band can be defined as the quotient of the plane $\mathbb{R}^2$ by the identification $(x+1,-y)\simeq(x,y)$.

2. Calculate the de Rham cohomology of the Möbius band from this definition, including the accurate description of all relevant spaces (note that the Möbius band is non-compact).

Problem 4. Let $\mathbb{D}$ be the unit disk $\{|z|<1\}\subseteq\mathbb{C}$ with the Riemannian metric

$$\rho=\frac{2|\mathrm{d}z|}{1-|z|^2}.$$

1. How do you understand this definition?
2. Show that the only *invertible* holomorphic map $f\colon\mathbb{D}\to\mathbb{D}$ which fixes the origin, is the rotation $f(z)=cz$, $|c|=1$. *Hint.* This is the Schwarz lemma from Complex Analysis in disguise, which in turn follows from the maximum modulus principle applied to the holomorphic function $f(z)/z$. Show that the map

$$z\mapsto\frac{z-a}{1-\bar{a}z},\qquad a\in\mathbb{D},$$

is invertible holomorphic self-map of $\mathbb{D}$. We denote by G the group of holomorphic self-maps of $\mathbb{D}$. 3. Show that G consists of isometries of ρ: any $g\in G$ preserves ρ.

4. Show that each transformation from G is *circular*: it sends circles in $\mathbb{C}$ to circles, if lines are considered as particular case of the circles.
5. Prove that the shortest path from any point $a\in\mathbb{D}$ to the origin is the segment of the corresponding radius. Prove that the diameters are geodesic curves for ρ.
6. Using symmetries from G, find and describe other geodesics, not passing through the origin. Describe all geodesics passing through a point $0\neq a\in\mathbb{D}$. Prove that for any two points $a\neq b\in\mathbb{D}$ there exists at least one circle that passes through a and b and is orthogonal to the unit circle $\{|z|=1\}$, called the *absolut*.

7. Let $\mathbb{H} = \{\operatorname{Im} z > 0\} \subset \mathbb{C}$ be the upper half-plane and

$$\rho' = \frac{|\mathrm{d}z|}{\operatorname{Im} z}$$

a Riemannian metric on it. Show that $\mathbb{H}$ and $\mathbb{D}$ are biholomorphically equivalent, and each conformal map $f\colon \mathbb{D} \to \mathbb{H}$ is an isometry conjugating ρ' with ρ. Describe the geodesic curves on $\mathbb{H}$.

8. Show that the Gauss curvature of the metric ρ' is equal to -1. *Hint.* The word *show* can be (only here) understood literally: if you can find these statements in a textbook, show me where.

9*. Suggest a Riemannian manifold with constant negative curvature $-\frac{1}{R^2}$, $1 \neq R$.

Problem 5. The complex projective line $\mathbb{C}P^1$ is formally defined as the quotient space $\mathbb{C}^2 \smallsetminus \{0\}$ by $\mathbb{C} \smallsetminus \{0\}$, that is, two points (x, y) and (u, v) are identified, if and only if there exists $\lambda \neq 0$ such that $(x, y) = \lambda(u, v)$. In the same way the projective plane $\mathbb{C}P^2$ is defined as the quotient $\mathbb{C}^3 \smallsetminus \{0\}/\mathbb{C} \smallsetminus \{0\}$.

1. Construct an atlas which makes $\mathbb{C}P^1$ into a holomorphic 1-dimensional manifold (holomorphic curve). Do the same for $\mathbb{C}P^2$. Show that $\mathbb{C}P^1 = \mathbb{C}^1 \sqcup \{*\}$, that is, the projective line can be obtained by adding a single point "at infinity" to the complex plane, which makes it topologically a sphere (*Riemann sphere*). Show $\mathbb{C}P^2 = \mathbb{C}^2 \sqcup \mathbb{C}P^1$, that is, the projective plane can be obtained by adding to the affine plane $\mathbb{C}^2$ the "line at infinity" $\mathbb{C}P^1$. *Hint.* The word *show* can be (only here) understood literally: if you can find these statements in a textbook, show me where.

2. Construct a nontrivial holomorphic vector field on $\mathbb{C}P^1$.

3. Construct a nonvanishing holomorphic vector field on $\mathbb{C}P^1$ or explain why it is impossible. *Hint*: what can be said about the sum total of the order of zeros and order of poles of any meromorphic function on $\mathbb{C}P^1$?

4. Construct a holomorphic 1-form on $\mathbb{C}P^1$ or explain why this is impossible.

5. Consider the circle $C = \{x^2 + y^2 = 1\} \subset \mathbb{R}^2$. Show that the differential 1-form $\omega = \frac{\mathrm{d}x}{y}$ is smooth on C.

6. Construct a rational map $U\colon \mathbb{C}P^1 \to \mathbb{C}P^2$ such that $U|_{\mathbb{R}} \subset C$ (this map is called *rational uniformization*). Show how this map allows to find all rational points on the circle $\mathbb{C}$ and all Pythagorean triples of natural numbers.

7. Compute the pullback $U^*\omega$ and show that it is meromorphic. Find its singular points and explain their location. *Hint.* A polynomial

is holomorphic on $\mathbb{C}^2$ but has singularities on the infinite line after extension on $\mathbb{C}P^2$.

8. Consider the algebraic curve $E = \{y^2 = p(x)\} \subseteq \mathbb{R}^2$, where p is a real polynomial of degree 3 with simple roots. Prove that the 1-form ω is holomorphic on the projective compactification of its complexification $\overline{E^{\mathbb{C}}} \subset \mathbb{C}P^2$. *Hint.* Where the troubles can come from? see the previous question.

9*. Find sufficient conditions guaranteeing that the 1-form $x^k \frac{\mathrm{d}x}{y}$ is holomorphic on the *hyperelliptic curve* defined by the equation $\{y^2 = p(x)\} \subseteq \mathbb{C}P^2$ with $\deg p = n$.