EXPONENT FUNCTION

Notation:

(1) $\mathbb{N}$ is the set of **positive** integer numbers $(1, 2, ...)$.
(2) $\mathbb{Z}$ is the set of integer numbers.
(3) $\mathbb{R}$ is the set of real numbers.
(4) For an integer $n \geq 0$, $n! = 1 \cdot 2 \cdot ... \cdot n$ (n factorial).

Definitions:

(1) A **series** is a formal infinite sum $a_0 + a_1 + a_2 + ... + a_n + ...$, where $\{a_n\}_{n \geq 0}$ is a sequence (for example, of real numbers). The **partial sums** of a series is the sequence $S_0 := a_0$, $S_1 := a_0 + a_1$, ..., $S_n := a_0 + a_1 + ... + a_n$.

We say that a series $a_0 + a_1 + a_2 + ... + a_n + ...$ **converges to a number** A if the sequence of partial sums $\{S_n\}_{n \geq 0}$ converges to A (we then denote $A = a_0 + a_1 + a_2 + ... + a_n + ...$ for short).

We say that a series $a_0 + a_1 + a_2 + ... + a_n + ...$ **converges absolutely** if the series $|a_0| + |a_1| + |a_2| + ... + |a_n| + ...$ converges.

Problem 1. In this problem, you are NOT allowed to use any information about exponent and logarithm functions. We want to define the exponent function using ONLY the operations of addition, subtraction, multiplication and division of real numbers.

(1) Prove that if a series $a_0 + a_1 + a_2 + ... + a_n + ...$ converges **absolutely**, then it converges (i.e. the sequence of partial sums has a limit).

Instructions: Let A be the limit of the series $|a_0| + |a_1| + |a_2| + ... + |a_n| + ...$ Let $\{S_n\}_{n \geq 0}$ be the sequence of partial sums of the series $a_0 + a_1 + a_2 + ... + a_n + ...$ Prove that the sequence $\{S_n\}_{n \geq 0}$ is bounded - i.e. that $|S_n| \leq A$. Use this to prove that $\{S_n\}_{n \geq 0}$ has a converging subsequence $\{S_{n_k}\}_{k \geq 0}$, which converges to a limit S. Now show that $\lim_{n \to \infty} S_n = S$.

(2) Let $x \in \mathbb{R}, x > 0$, and let $n \in \mathbb{Z}, n \geq 0$. Put $a := \lfloor 2x \rfloor + 1$. Prove that $1 + x + \frac{x^2}{2!} + ... + \frac{x^n}{n!} \leq a^a + \frac{a^a}{2} + ... + \frac{a^a}{2^n}$

(3) Prove that for every $x \in \mathbb{R}$, the series $1 + x + \frac{x^2}{2!} + ... + \frac{x^n}{n!} + ...$ converges.

Hint: Prove that for every $x \in \mathbb{R}, x \geq 0$, the partial sums of the series is a sequence which is bounded by $2(\lfloor 2x \rfloor + 1)^{\lfloor 2x \rfloor + 1}$. Use this to prove that the series $1 + x + \frac{x^2}{2!} + ... + \frac{x^n}{n!} + ...$ converges absolutely for **every** $x \in \mathbb{R}$.

(4) We define the exponent function $exp : \mathbb{R} \to \mathbb{R}$ as follows: $exp(x) := 1 + x + \frac{x^2}{2!} + ... + \frac{x^n}{n!} + ...$ Prove that $exp(x + y) = exp(x)exp(y)$ for any $x, y \in \mathbb{R}$.

Instructions: For each $n \geq 0$, denote $a_n := 1 + x + \frac{x^2}{2!} + ... + \frac{x^n}{n!}$, $b_n := 1 + y + \frac{y^2}{2!} + ... + \frac{y^n}{n!}$, $c_n := 1 + xy + \frac{(xy)^2}{2!} + ... + \frac{(xy)^n}{n!}$. For $x, y \geq 0$, prove that $a_n b_n \leq c_{2n} \leq a_{2n} b_{2n}$ and then use the Sandwich theorem to complete the proof. Give a similar proof when x or y is negative.

(5) Prove that the function exp is strictly increasing, i.e. for any $x < y$, prove that $exp(x) < exp(y)$. Remember that x, y might be negative! *Instructions:* Prove that $exp(x) > 1$ for any $x > 0$, and use this fact to prove the required statement.

(6) We denote by e the value $exp(x)$. Check that $e > 1$, and prove that for any rational number $\frac{m}{n} \in \mathbb{Q}$, $n > 0$, we have: $exp(\frac{m}{n})^n = e^m$.

(7) Prove that $exp(x) > 0$ for any $x \in \mathbb{R}$.

(8) Prove that the function exp is continuous at point $x = 0$. *Hint:* Prove that for every x, $|exp(x) - 1| \leq |x| \cdot exp(|x|)$, and conclude that for every $x \in (-1, 1)$, we have: $|exp(x) - 1| \leq |x| \cdot e$.

(9) Prove that the function exp is continuous (at every point in $\mathbb{R}$).

(10) Prove that $exp(\mathbb{R}) = (0, \infty)$, i.e. prove that for any $y \in (0, \infty)$, there exists $x \in \mathbb{R}$ such that $exp(x) = y$. *Hint:* Use Intermediate Value Theorem.

(11) Prove that there **exists** an inverse to the map $exp : \mathbb{R} \to (0, +\infty)$. We denote this function by $\ln(x) : (0, +\infty) \to \mathbb{R}$. Prove that the function $\ln(x)$ is continuous in the ray $(0, +\infty)$.

(12) Prove that the function exp is differentiable at every point in $\mathbb{R}$, and $exp'(a) = exp(a)$.