Exercise 25.13. Prove that a quadratic planar vector field with three invariant lines in general position admits a Darbouxian first integral.

Exercise 25.14. Prove that the bounds in Theorems 25.16 and 25.15 are sharp: there exist foliations from the class $\mathcal{B}_r$ which have algebraic invariant curve of degree $r+1$ and smooth algebraic curve of degree r.

26. Perturbations of Hamiltonian vector fields and zeros of Abelian integrals

Limit cycles are very difficult to track in general. The problem can be considerably simplified by *localization* in the phase space and/or parameters. For instance, restricting the domain in the phase plane to a neighborhood of an elliptic singular point allows us to track small amplitude limit cycles, as explained in §12. Another possibility implicitly explored in §24, is the study of limit cycles near separatrix polygons (*polycycles*).

One of the most powerful methods of analysis in general is localization in the parameter space: starting from an object with known simple properties, investigate what happens after small perturbation. In application to the study of vector fields, appearance and disappearance of limit cycles goes by the name of *bifurcation*.

In this section we consider *bifurcations of limit cycles from nonisolated periodic orbits*. The number and location of these cycles in the most important cases is determined by zeros of a special class of functions, Abelian integrals. Recall that Abelian integrals are integrals of rational 1-forms over cycles on algebraic curves, and if considered as functions of the parameters (coefficients of polynomials defining the curves), they are transcendental functions of several complex variables.

We study the algebraic and topological structure of Abelian integrals, proving several fundamental results that are widely used but not yet available in a complete and elementary exposition. The central algebraic result is description of the *module* of Abelian integrals over the ring of polynomials and explicit computation of the basis of this module. The topological study allows us to compute the monodromy group of continuous branches of Abelian integrals. Finally we bring together the two theories and derive a *Picard–Fuchs system* of linear ordinary differential equations for Abelian integrals, establish the type of its singularities and almost irreducibility of its monodromy. This opens the way to apply the "linear" tools developed in Chapter III to investigation of bifurcations of nonlinear systems (though this way remains unexplored in the book[5]).

[5] For further developments in this direction see the publications [**IY96, NY99, NY01, NY03, NY04, Yak05, Yak06**]. An alternative approach can be found in [**GI06, GI07**].

26A. Poincaré–Pontryagin criterion and generalizations. If γ is a closed (periodic) *nonisolated* orbit of a real analytic vector field F, then it must be an identical cycle by Theorem 9.12: some sufficiently narrow annulus-like neighborhood U of γ is entirely filled by closed orbits of F. In this case F is *analytically* integrable in U: there exists a real analytic function $f: U \to \mathbb{R}$ without critical points, such that $Ff = 0$ (Problem 26.1). The Pfaffian equation for the foliation takes the form $\{df = 0\}$.

Let $\varepsilon \in (\mathbb{R}^1, 0)$ be a small real parameter and f a real analytic function without critical points as above. Consider a real analytic perturbation of the initial integrable foliation $\{df = 0\}$, written in the Pfaffian form as

$$df + \varepsilon\omega = 0, \qquad \omega \in \Lambda^1(U), \quad \varepsilon \in (\mathbb{R}^1, 0), \tag{26.1}$$

where $\omega \in \Lambda^1(U)$ is a real analytic 1-form. Denote by

$$\Delta = \Delta_\gamma : (\mathbb{R}^1, 0) \times (\mathbb{R}^1, 0) \to (\mathbb{R}^1, 0), \qquad (z, \varepsilon) \mapsto \Delta(z, \varepsilon),$$

the holonomy map of the cycle γ considered as a function of the parameter ε. Since all elements of the construction are real analytic, Δ can be expanded in the converging series,

$$\Delta(z, \varepsilon) = z + \varepsilon I_1(z) + \cdots + \varepsilon^k I_k(z) + \cdots, \tag{26.2}$$

where $I_k(z)$ are real analytic functions defined in some common neighborhood of the origin $z = 0$. Since the case $\varepsilon = 0$ corresponds to the integrable system, the term I_0 is absent in (26.2).

The first not identically zero function in the sequence $I_1, I_2, \dots,$ plays a special role.

Proposition 26.1. *Assume that the first nonzero function $I_k(z)$ for some $k \geqslant 1$ has n isolated zeros (counted with their multiplicities) in the closed interval $\{|z| \leqslant \rho\}$. Then there exists a small positive value $r > 0$ such that the foliation (26.1) in $\{|z| < \rho\} \subset U$ has no more than n limit cycles for all $|\varepsilon| < r$.*

Proof. Limit cycles correspond to the roots of the equation $\Delta(z, \varepsilon) - z = 0$. In the assumptions of the proposition the left hand side is divisible by ε^k: $\Delta(z, \varepsilon) - z = \varepsilon^k I'(z, \varepsilon)$. The number of isolated roots of the real analytic function $I'(z, \varepsilon) = I_k(z) + \varepsilon I_{k+1}(z) + \cdots$ for all sufficiently small ε does not exceed the total number of roots of its limit $I_k(z) = \lim_{\varepsilon \to 0} I'(z)$. □

Remark 26.2. The number of geometrically distinct zeros of the first nonzero function I_k can provide also a *lower* bound for the number of limit cycles, if the former are all of an odd multiplicity (e.g., all simple). If the first nonzero function I_k has a real root of an *odd* order, then the Poincaré function has *at least one* real root which is a limit cycle (this obviously follows from the intermediate value theorem). Other roots in general may be complex and do not correspond to limit cycles. For roots of I_k of an even order *all* roots of the displacement function may escape to the nonreal domain and do not manifest themselves as limit cycles.

The analytic expression of the first variation for the perturbation (26.1) is very simple.

Theorem 26.3 (Poincaré [**Poi90**], Pontryagin [**Pon34**]).

$$I_1(z) = -\oint_{\{f=z\}} \omega. \tag{26.3}$$

If the integral (26.3) is not identically zero, it may have only finitely many zeros on the cross-section τ. By Proposition 26.1, this number is an upper bound for the number of limit cycles of the perturbed foliation (26.1) for all sufficiently small values of the parameter ε.

Proof. Denote by $\gamma_{z,\varepsilon}$ the arc of an integral curve of the perturbed foliation between the point with the coordinate z on the cross-section τ and the next intersection with τ. By the choice of the chart $z = f|_\tau$ and the definition of the displacement,

$$\Delta(z,\varepsilon) - z = \int_{\gamma_{z,\varepsilon}} df = -\varepsilon \int_{\gamma_{z,\varepsilon}} \omega.$$

The last equality holds since $df + \varepsilon\omega$ vanishes identically on $\gamma_{z,\varepsilon}$ for any z. As $\varepsilon \to 0$, the arc $\gamma_{z,\varepsilon}$ tends uniformly in the C^1-sense to the closed curve $\gamma_{z,0} = \{f = z\}$. Hence the integral $\int_{\gamma_{z,\varepsilon}} \omega$ converges to the integral in (26.3). □

26B. Higher variations of the holonomy. If the Poincaré integral (26.3) vanishes identically, the higher variations I_k, $k = 2, 3, \dots$ should be computed until either a not identically vanishing variation is found, or for some reason it becomes clear that the family (26.1) entirely consists of integrable foliations for all small values of ε.

We describe an analytic procedure expressing the first nonzero function $I_k(z)$ as an integral of a certain analytic 1-form ω_k along the level ovals $\{f = z\} \subset U$. To describe this procedure, we need the following simple analytic observation.

Consider a domain $U \subset \mathbb{R}^2$ and a real analytic function $f\colon U \to \mathbb{R}$ without critical points in it.

Definition 26.4. A real analytic 1-form $\alpha \in \Lambda^1(U)$ is relatively exact with respect to the integrable foliation $\mathcal{F} = \{df = 0\}$ in a domain U, if

$$\alpha = h\,df + dg, \qquad h, g \in \mathcal{O}(U), \tag{26.4}$$

with two functions g, h real analytic in U.

The integral of a relatively exact form α along any closed oval on any level curve $\{f = z\} \subset U$ is obviously zero:

$$\forall \text{ oval } \delta \subseteq \{f = z\} \qquad \oint_\delta \alpha = 0. \tag{26.5}$$

The inverse assertion (and especially the complexification thereof) is considerably more delicate. It holds true only under some additional topological assumptions; see §26**D** below. The simplest case, however, is rather easy.

Lemma 26.5. *If U is the topological annulus formed by ovals of the level curves $\{f = z\}$ transversal to a global cross-section τ and a form $\alpha \in \Lambda^1(U)$ satisfies the condition* (26.5)*, then α is relatively exact in U.*

Proof. For any $x \in U$ denote by $\gamma(x)$ an oriented arc of the level curve passing through x between x and the point of its intersection with τ. This arc is defined modulo an integer multiple of the loop (oval) $\delta = \{f = z\}$, $z = f(x)$, yet because of the condition (26.5), the integral $g(x) = \int_{\gamma(x)} \alpha$ is a well-defined analytic function in U. By construction, the forms α and dg take the same values on each vector tangent to any level curve $\{f = z\} \subset U$, i.e., the difference $\alpha - dg$ at each point is proportional to df. Since df never vanishes in U, the proportionality coefficient is a real analytic function: $\alpha - dg = h\, df$ for some $h \in \mathcal{O}(U)$. □

Remark 26.6. The representation (26.4) is not unique. One can replace $g(x)$ by $g(x) + u(f(x))$ with an arbitrary function u.

To compute the first function I_i which does not vanish identically, we construct inductively, using the representation (26.4), the sequence of real analytic 1-forms $\omega_1, \omega_2, \dots \in \Lambda^1(U)$ as follows.

1°. (*Base of induction*). $\omega_1 = \omega$ is the perturbation form from (26.1).

2°. (*Induction step*). If the forms $\omega_1, \dots, \omega_j$ are already constructed and turned out to be relatively exact, then by Lemma 26.5, $\omega_j = h_j\, df + dg_j$. In this case we define

$$\omega_{j+1} = -h_j\, \omega. \tag{26.6}$$

Theorem 26.7. *If ω_k, $k \geqslant 2$, is the first not relatively exact 1-form in the sequence $\omega_1, \dots, \omega_{k-1}, \omega_k$ constructed inductively by* (26.6)*, then*

$$I_k(z) = -\oint_{\{f=z\}} \omega_k. \tag{26.7}$$

This theorem generalizes the Poincaré–Pontryagin Theorem 26.3. The algorithm of inductive construction of the forms ω_k, sometimes referred to as the *Françoise algorithm*, was independently suggested in [**Yak95**] and [**Fra96**], but probably was known much earlier.

Proof. Denote by U' the annulus U slit along the cross-section τ. This is a simply connected domain (curvilinear rectangle) foliated by the level curves of the function f, transversal to the two sides (denoted by τ_- and τ_+). Denote $\mathcal{F}_\varepsilon$ the foliation defined by the Pfaffian equation (26.1) in U'.

1. Let $u = u(x, \varepsilon) = u_\varepsilon(x)$ be the first integral of the foliation $\mathcal{F}_\varepsilon$ in U',

$$(df + \varepsilon\omega) \wedge du_\varepsilon \equiv 0, \tag{26.8}$$

which for $\varepsilon = 0$ coincides with f and analytically depends on the parameter ε; such an integral exists because the topology of the foliation $\mathcal{F}_\varepsilon$ in the slit annulus U' is trivial. (This integral is not uniquely defined for $\varepsilon \neq 0$.)

Denote by $z_\varepsilon = u_\varepsilon|_{\tau_+}$ the restriction of u_ε on the "terminal" side τ_+ of the cross-section τ. Being a small analytic perturbation of the chart $z = u_0|_\tau$, z_ε is also an analytic chart on τ. Then the restriction of the same solution u_ε on the "initial" side $u_\varepsilon|_{\tau_-}$ is the numeric value of the holonomy map $\Delta(\cdot, \varepsilon)$ related to the chart z_ε. Indeed, since u_ε is constant along integral curves, for points on τ_- it yields the value of the chart z_ε at the moment of the next hit.

In other words, the displacement function $\Delta(z_\varepsilon, \varepsilon) - z_\varepsilon$ related to the chart z_ε, is given by the difference $u_\varepsilon|_{\tau_-} - u_\varepsilon|_{\tau_+}$ of the first integral u_ε. Since level curves of f in U' are oriented in the direction from τ_- to τ_+, it is more natural to compute the *negative* of this expression, the difference

$$u_\varepsilon\Big|_{\tau_-}^{\tau_+} = -\big(\Delta(z_\varepsilon, \varepsilon) - z_\varepsilon\big). \tag{26.9}$$

2. The convenience of expressing the displacement function in terms of solutions of the partial differential equation (26.8) stems from the *linearity* of the latter. In particular, one can look for its solution in terms of the converging series,

$$u_\varepsilon = f + \varepsilon u_1 + \varepsilon^2 u_2 + \cdots, \tag{26.10}$$

where u_k are real analytic functions in the slit annulus U'. Substituting this series into the equation (26.8), we obtain the following system of equations on the respective components.

$$\begin{aligned} \omega \wedge df + df \wedge du_1 &= 0, \\ \omega \wedge du_1 + df \wedge du_2 &= 0, \\ \cdots\cdots\cdots\cdots\cdots\cdots & \\ \omega \wedge du_{k-1} + df \wedge du_k &= 0. \end{aligned} \tag{26.11}$$

By assumption, the forms $\omega_1, \dots, \omega_{k-1}$ are all relatively closed, hence they admit representations

$$\omega_j = h_j\, df + dg_j, \qquad j = 1, \dots, k-1.$$

We claim that the functions $u_j = g_j$ satisfy the first $k-1$ equations of system (26.11). Indeed, direct substitution yields for all $j = 1, \dots, k-2$

$$\begin{aligned}\omega \wedge du_j + df \wedge du_{j+1} &= \omega \wedge (\omega_j - h_j\, df) + df \wedge (\omega_{j+1} - h_{j+1}\, df) \\ &= -h_j \omega \wedge df + df \wedge \omega_{j+1} = df \wedge (\omega_{j+1} + h_j \omega) = 0.\end{aligned}$$

The fact that the first $k-1$ components of the solution (26.10) are well-defined functions in the *nonslit* annulus U means that their contribution to the difference (26.9) is zero, $u_k\big|_{\tau_-}^{\tau_+} \equiv 0$, and all Melnikov functions $I_1, \dots, I_{k-1}$ are vanishing identically.

3. The kth equation of the system (26.11) can be used to determine the component u_k. The same computation as above reduces this equation to the form

$$\begin{aligned}0 &= \omega \wedge (-h_{k-1}\, df) + df \wedge du_k = df \wedge (du_k + h_{k-1}\omega) \\ &= df \wedge (du_k - \omega_k).\end{aligned}$$

This means that the 1-form $du_k - \omega_k$ vanishes on all level curves $f = \text{const}$, i.e., that u_k can be restored as the primitive along these curves,

$$u_k(x) = \int_{\tau_-}^{x} \omega_k,$$

where the path of integration is the arc $\gamma(x)$ of the level curve, connecting an appropriate point on the slit τ_- with the variable point $x \in U'$. The difference (increment) of u_k from τ_- to τ_+ is then equal to the integral along the entire oval,

$$u_k(z_\varepsilon)\Big|_{\tau_-}^{\tau_+} = \oint_{\{f = z_\varepsilon\}} \omega_k.$$

From (26.9) we conclude that

$$\Delta(z_\varepsilon, \varepsilon) - z_\varepsilon = -\varepsilon^k u_k(z_\varepsilon)\Big|_{\tau_-}^{\tau_+} + O(\varepsilon^{k+1}) = -\varepsilon^k \oint_{\{f = z_\varepsilon\}} \omega_k + O(\varepsilon^{k+1}).$$

As $\varepsilon \to 0$, the chart z_ε converges uniformly to the chart $z = f|_\tau$, and we obtain the assertion of the theorem. □

Remark 26.8. The functions h_k playing the key role in the inductive construction, can also be restored as integrals of appropriate forms. Indeed, they satisfy the equations $d\omega_k = dh_k \wedge df$ and hence can be restored as the primitives

$$h_k(\cdot) = \int^{\bullet} \frac{d\omega_k}{df} \qquad (26.12)$$

along the level curves of f (the form $\frac{d\omega_k}{df}$ is the Gelfand–Leray derivative of ω; see §26**G** below). If h_k is any solution of the equation $d\omega_k = dh_k \wedge df$, then the form $\omega_k - h_k\, df$ is closed and, under the condition that periods of ω_k are all zero, is exact in U, since its integral over the oval $f = \text{const}$ generating the homology group of U, is zero.

The representation (26.12) allows us to write down all forms ω_k as *iterated integrals* along the level curves of f. The details can be found in [**Gav05**].

Note also that the above approach can be almost literally applied to a perturbation more general than (26.1),

$$df + \varepsilon\theta_1 + \varepsilon^2\theta_2 + \cdots = 0, \qquad \theta_1, \theta_2, \dots \in \Lambda^1(U). \tag{26.13}$$

The system (26.11) becomes nonhomogeneous but still retains the triangular form allowing for an explicit solution.

26C. Infinitesimal Hilbert's sixteenth problem. Proposition 26.1 and Theorems 26.3 and 26.7 indicate that in order to bound the number of limit cycles which appear by polynomial perturbation $\mathcal{F}_\varepsilon = \{\theta + \varepsilon\omega = 0\}$ of a polynomial *integrable* foliation $\mathcal{F}_0 = \{\theta = 0\}$ on $\mathbb{R}P^2$, it is necessary to estimate the number of zeros of integrals of the rational 1-form ω over the nonisolated ovals of the unperturbed foliation $\mathcal{F}_0$. The problem of finding an explicit upper bound for this number in terms of the degrees of θ and ω is referred to by numerous names: infinitesimal Hilbert problem, relaxed Hilbert problem, Hilbert–Arnold problem, tangential Hilbert problem, *etc.* In the above formulation the problem appeared between the lines in [**Ily69**] and since then repeatedly mentioned by Arnold in his seminar; see [**Arn04**].

However, when the initial foliation is defined by a *closed rational* 1-form θ, the first integral f can be *nonalgebraic* (cf. with §25$\mathbf{G}_1$). Limit cycles can also be born from *separatrix polygons* of $\mathcal{F}_0$ rather than from ovals, in which case an additional analysis is required (Proposition 26.1 does not apply in this case). Finally, the problem which turns out to be transcendentally difficult, is to determine *how many identically zero Melnikov functions should be computed before one can guarantee that the perturbation in fact preserves the integrability.* Even in the most simple case where the foliation is by circles, $f(x,y) = x^2 + y^2$, so that all integrals I_k are in fact polynomial functions of z, this question is open (the so-called *Poincaré problem*). All these difficulties force us to concentrate on the first really nontrivial case of Abelian integrals which appear as follows.

Consider the important class of *Hamiltonian foliations* defined by *exact polynomial* form df with a real polynomial $f \in \mathbb{R}[x,y]$ of some degree $\deg f = n+1$. If the perturbation form ω in (26.1) is also polynomial, then we arrive at the following restricted formulation of a problem continuing the series of Hilbert-type problems from §24**A**.

Problem IX (infinitesimal version of Hilbert's sixteenth problem)**.** *Find an upper bound for the number of* isolated *zeros of the integral* $I(z) = \oint_{\{f=z\}} \omega$ *of a polynomial* 1*-form* ω *over the algebraic ovals* $\{f = z\}$ *in terms of the degrees* $\deg df$ *and* $\deg \omega$*.*

Since all other settings are practically unexplored, we will refer to this restricted formulation as *the* infinitesimal Hilbert problem.

Definition 26.9. A (complete) Abelian integral is the integral $\oint_\delta \omega$ of a rational 1-form ω over an oval of an algebraic curve $\delta \subseteq \{f = 0\}$. This integral depends on the coefficients of the form $\omega \in \Lambda^1[x, y]$ and of the polynomial $f \in \mathbb{R}[x, y]$ as the parameters.

In most cases we will fix the form ω and all coefficients of f except for the free term, and consider the corresponding Abelian integral as the function of only one parameter,

$$I_{f,\omega}(z) = \oint_{\{f=z\}} \omega. \tag{26.14}$$

The infinitesimal Hilbert problem as it appears above, is the *problem on the maximal possible number of real isolated zeros of the Abelian integral* (26.14).

Note that the function $I_{f,\omega}(z)$ is in general *multivalued*, since the real level curve of f may consist of several ovals (besides noncompact components). However, any compact real oval $\delta \subseteq \{f = z_0\}$ can be continuously deformed to a uniquely defined compact oval on all sufficiently close level curves $\{f = z\}$, $z \in (\mathbb{R}, z_0)$. This allows us to define unambiguously *continuous real branches* of the Abelian integral (26.14). Simple arguments show that each continuous branch is real analytic in the interior of its domain (Problem 26.3).

26D. Relative cohomology and integrals: algebraic vs. analytic. The global algebraic nature of the infinitesimal Hilbert problem justifies introduction of a special algebraic language of *relative* cohomology. This language is parallel to the *de Rham cohomology* which describes the difference between closed and exact differential forms; see [**War83**].

26**D**$_1$. *Relative de Rham complex and its cohomology.* The condition (26.5) can be interpreted in cohomological terms as follows. Consider the de Rham complex[6]

$$0 \longrightarrow \Lambda^0(U) \xrightarrow{d} \Lambda^1(U) \xrightarrow{d} \Lambda^2(U) \xrightarrow{d} 0 \tag{26.15}$$

[6] In general, an algebraic complex is a chain of modules $A_0 \xrightarrow{d} A_1 \xrightarrow{d} A_2 \xrightarrow{d} A_3 \cdots$ with a derivation d whose square $d \circ d$ is zero.

formed by the modules $\Lambda^k = \Lambda^k(U)$ of real analytic k-forms in the domain U and the exterior derivative d (we deal only with the 2-dimensional domain U, but all constructions can be instantly generalized for an arbitrary dimension). The exterior derivative d takes the submodule $df \wedge \Lambda^{k-1} \subseteq \Lambda^k$ into $df \wedge \Lambda^k \subseteq \Lambda^{k+1}$, which means that d descends to an operator (also denoted by d) between the *modules of relative k-forms*, the quotient modules $\Lambda^k_f(U) = \Lambda^k(U)/df \wedge \Lambda^{k-1}(U)$. Passing to the quotients transforms the de Rham complex (26.15) into the *relative de Rham complex*

$$0 \longrightarrow \Lambda^0(U) \xrightarrow{d} \Lambda^1_f(U) \xrightarrow{d} \Lambda^2_f(U) \xrightarrow{d} 0. \qquad (26.16)$$

The cohomology of this complex, the quotients $\operatorname{Ker} d/\operatorname{Im} d$, is called the *relative cohomology* $H^k_f(U)$, $k = 0, 1, \dots, n$ (to make the term precise, one has to specify the ring of functions—polynomial, real analytic, smooth, *etc.*; see below).

The zero relative cohomology module $H^0_f(U) = \{g \in \mathcal{O}(U) : dg = h\, df\}$ can be identified with functions constant along the level curves of f. If f has no critical points in U, *any* 2-form is divisible by df. To prove that, it is sufficient to construct just one area form (nonvanishing 2-form) divisible by df; any other 2-form will then be proportional to it hence also divisible by df. If $\theta\colon U \to \mathbb{R} \bmod 2\pi$ is the cyclic variable ("polar angle") along the ovals $f = \text{const}$, then the required area form is $df \wedge d\theta$. Thus in the absence of critical points of f, $\Lambda^2_f(U) = 0$ and hence $H^2_f(U) = 0$.

The only dimension when the relative cohomology is nontrivial, is 1. The definition of relative exactness was given earlier (Definition 26.4). On the other hand, since $\Lambda^2_f(U) = 0$, any 1-form is relatively closed. In terms of the relative cohomology, Lemma 26.5 asserts that the *period map*

$$H^1_f(U) \to H^0_f(U), \qquad \alpha \mapsto g(x) = \oint_{\gamma \ni x} \alpha$$

(the integral is taken over the oval $f = \text{const}$ passing through the point x), is an isomorphism.

Remark 26.10. The notion of relative cohomology can be defined for any *closed* (not necessarily exact) form $\theta \in \Lambda^1$ as the cohomology of the complex $\Lambda^k_\theta(U) = \Lambda^k(U)/\theta \wedge \Lambda^{k-1}(U)$. However, in this case the analysis is considerably more subtle; see [**BC93**].

The construction of relative cohomology depends on the base ring, from which the function f and the coefficients of the form α are taken. Thus far we were dealing with real analytic forms and functions in an annulus. However, when dealing with Abelian integrals, it is natural to assume that the base ring is $\mathbb{R}[x, y]$ or $\mathbb{C}[x, y]$, and all forms are also polynomial.

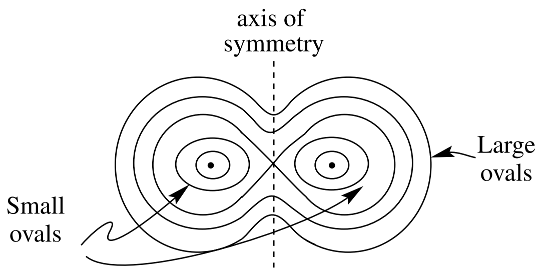

Figure V.1. Three continuous families of ovals

The straightforward generalization of Lemma 26.5 for *polynomial* rather than real analytic 1-forms fails. The integral of a polynomial 1-form ω over a continuous family of ovals on the level curves of a polynomial f may vanish identically, yet the form ω may not admit the representation (26.4) with *polynomial* g and h.

Example 26.11. Consider the symmetric polynomial $f(x,y) = y^2 - x^2 + x^4$. The real level curves $\{f = z\}$ are empty for $z < -\frac{1}{2}$, carry two "small" ovals for $z \in (-\frac{1}{2}, 0)$ and only one "large" oval for $z > 0$; see Fig. V.1. The large ovals are all symmetric with respect to both axes, while the "small" ovals are symmetric only in the y-axis.

Integral of the 1-form $\omega = x^2\,dy$ over the family of "large" ovals vanishes identically, since the integrals over the two parts in the half-planes $\{x > 0\}$ and $\{x < 0\}$ mutually cancel each other.

Yet the form ω cannot be represented under the form (26.4) with *polynomial* g and h; if this were the case, then the integral of ω over any of the two families of "small" cycles would also vanish identically. Yet $d\omega = -2x\,dx \wedge dy$ keeps constant sign in each half-plane $\{\pm x > 0\}$, hence the integral over each of these two families is nonzero.

The explanation of this phenomenon lies in the fact that after analytic continuation Abelian integrals become multivalued functions ramified over a certain finite set. One branch can be identical zero, while others not. However, if *all complex* branches vanish identically, the validity of the assertion is restored, as asserts Theorem 26.13 below.

26$\mathbf{D}_2$. *Analytic relative cohomology.* To achieve a proper complexification of Abelian integrals, consider the real polynomial $f \in \mathbb{C}[x,y]$ as a complex function $f \colon \mathbb{C}^2 \to \mathbb{C}$, and denote by $L_z = f^{-1}(z) \subset \mathbb{C}^2$ the *complex affine* level curves. These curves taken together form the (singular holomorphic) Hamiltonian foliation $\mathcal{F} = \{df = 0\}$ on $\mathbb{P}^2$.

The projective compactification of the leaves $\overline{L_z} \in \mathbb{P}^2$ may be singular or not; singular curves correspond to isolated values of z from some *critical locus* $\Sigma \subset \mathbb{C}$.

Any polynomial 1-form ω restricted on a nonsingular curve L_z is a closed form (for reasons of dimension) with the poles at infinity. The restriction $\omega|_{L_z} \in \Lambda^1(L_z)$ on L_z is exact, if and only if the integral of ω along any cycle δ on L_z is zero.

Definition 26.12. A rational 1-form is *analytically relatively exact* with respect to an integrable (Hamiltonian) foliation $\mathcal{F} = \{df = 0\}$ on $\mathbb{P}^2$ with $f \in \mathbb{C}[x, y]$, if the integral of ω along any cycle $\delta \in H_1(L, \mathbb{Z})$ on any leaf $L \in \mathcal{F}$ is zero,

$$\forall L \in \mathcal{F},\ \forall \delta \in H_1(L, \mathbb{Z}), \quad \oint_\delta \alpha = 0. \tag{26.17}$$

Clearly, any form that is *algebraically relatively exact*, i.e., representable under the form

$$\omega = h\, df + dg, \qquad h, g \in \mathbb{C}[x, y], \tag{26.18}$$

(cf. with Definition 26.4), is also analytically relatively exact with respect to the Hamiltonian foliation $\mathcal{F} = \{df = 0\}$. The inverse statement, a genuine algebraic counterpart of Lemma 26.5, is the following theorem.

Theorem 26.13 (Yu. Ilyashenko [**Ily69**], L. Gavrilov [**Gav98**]). *Assume that the polynomial $f \in \mathbb{C}[x, y]$ satisfies the following two conditions*:

(1) *all affine level curves $L_z = f^{-1}(z) \subset \mathbb{C}^2$ are connected, and*

(2) *all critical points of f in $\mathbb{C}^2$ are isolated.*

Then a polynomial 1-form ω is algebraically relatively exact with respect to the Hamiltonian foliation $\mathcal{F} = \{f = 0\}$ if and only if it is analytically relatively exact with respect to it.

In other words, for Hamiltonian foliations satisfying the assumptions of the theorem, any 1-form with zero periods on all leaves is representable under the form (26.18).

Proof. In one direction the theorem is trivial. To prove the other direction, denote by n the degree of f. Without loss of generality we may assume that the polynomial f restricted to the y-axis $Y = \{x = 0\}$ has the same degree n (this can always be achieved by a suitable affine change of variables x, y).

Each level curve L_z intersects Y by the same number of points $p_1(z), \dots, p_n(z)$ (every point is counted as many times as the multiplicity of the root of $f(0, y) - z$). For a point $(x, y) \in \mathbb{C}^2$ let $\gamma_j(x, y)$ be a path connecting this point with the jth point $p_j(z)$ on the intersection of the level curve L_z, $z = f(x,y)$, passing through it, with Y. Existence of such paths follows from the first assumption of the theorem.

The paths $\gamma_1(x,y),\dots,\gamma_n(x,y)$ are defined only modulo elements from $H_1(L_z,\mathbb{Z})$, but if the restriction $\omega|_{L_z}$ is exact, then the integrals $\int_{\gamma_j(x,y)}\omega$ are uniquely defined. The function

$$g(x,y) = \frac{1}{n}\sum_{j=1}^{n}\int_{\gamma_j(x,y)}\omega \tag{26.19}$$

is correctly defined on the complement to the union of critical level curves $S = f^{-1}(\Sigma) \subset \mathbb{C}^2$, since it does not depend on the (noninvariant) way of enumeration of the points $p_j(z)$ and the freedom in the choice of the paths $\gamma_j(x,y)$. Moreover, it is holomorphic on $\mathbb{C}^2 \smallsetminus S'$, where S' is the union of singular level curves S and the level curves tangent to the axis Y. Indeed, in this case one can choose the paths $\gamma_j(x,y)$ analytically depending on (x,y), say, as lifts on the level curves of some paths on the x-plane, connecting the origin $x = 0$ with the variable point x while avoiding the critical points of the projection $(x,y) \mapsto x$ restricted on L_z.

The function $g(x,y)$ remains bounded near the algebraic set $S' \subset \mathbb{C}^2$ of codimension 1; therefore, it extends analytically on S' as an entire function. Moreover, as (x,y) tends to infinity, both the length of the paths and the integrand in (26.19) grow at most polynomially in $|x| + |y|$, therefore the averaged primitive given by this expression, is a *polynomial*: $g \in \mathbb{C}[x,y]$.

The polynomial 1-form $\omega - dg$ by construction vanishes on all vectors tangent to the level curves. Since the form df does the same, we conclude that $\omega - dg = h\,df$, where h is a meromorphic function on $\mathbb{C}^2$ defined on the complement to the set of critical points where df vanishes. If this set is zero-dimensional (of codimension 2 in $\mathbb{C}^2$), as follows from the second assumption of the theorem, then h necessarily extends analytically to these points and hence is a *polynomial*, $h \in \mathbb{C}[x,y]$. The required representation is constructed. □

26**D**$_3$. *Bonnet theory.* Assumptions of Theorem 26.13 are rather nonrestrictive: the first is guaranteed automatically, if the Hamiltonian foliation $\mathcal{F}$ has only simple (in some sense) singularities on the infinite line, while the second assumption holds true for any square-free polynomial. If one of these assumptions is violated, the analytic relative exactness may not imply the algebraic one. Theorem 26.13 is in fact the particular case of a more general assertion concerning the relative cohomology.

Definition 26.14. The *Bonnet set* $\mathrm{Bs}(f)$ of a polynomial $f \in \mathbb{C}[x,y]$ is the set of values z such that the affine level curve $L_z = f^{-1}(z) \subset \mathbb{C}^2$ is either nonconnected or carries a nonisolated critical point of f.

In the assumptions of Theorem 26.13, the Bonnet set is empty.

Theorem 26.15 (P. Bonnet, 1999 [**Bon99, BD00**]). *Assume that the Bonnet set of a polynomial f is finite.*

Then for any polynomial 1-form which is exact on each level curve L_z, there exist a pair of polynomials $g, h \in \mathbb{C}[x,y]$ and a polynomial $b \in \mathbb{C}[z]$, nonvanishing outside the Bonnet set $\mathrm{Bs}(f)$*, such that*

$$(b \circ f)\,\omega = h\,df + dg. \tag{26.20}$$

Proof. For any $z \in \mathbb{C} \smallsetminus \mathrm{Bs}(f)$ the form $\omega|_{L_z}$ is exact. Since L_z is assumed connected, there exist a polynomial $g_z(x,y)$ such that $\omega - dg_z$ vanishes on all vectors tangent to L_z. This means that

$$\omega - dg_z = (f-z)\xi_z + u\,d(f-z) = (f-z)\theta_z, \qquad \theta_z \in \Lambda^1[x,y]$$

(we integrated by parts). The polynomial forms θ_z are defined for all values $z \in \mathbb{C} \smallsetminus \Sigma$ (uncountably many of them), whereas the number of different monomial 1-forms is countable. Therefore there must exist a linear dependence between the forms θ_z, involving only finitely many of them:

$$\sum_{j=1}^{m} \lambda_j \theta_{z_j} = 0, \qquad \text{for some } z_1, \dots, z_m \in \mathbb{C} \smallsetminus \Sigma. \tag{26.21}$$

The identity (26.21) can be rewritten as follows:

$$\omega \cdot \sum_{j=1}^{m} \frac{\lambda_j}{f - z_j} = \sum_{j=1}^{m} \frac{\lambda_j\,dg_j}{f - z_j},$$

or, after getting rid of the denominators,

$$B_0(f)\,\omega = \sum_j B_j(f)\,dg_j, \qquad B_0, B_1, \dots, B_m \in \mathbb{C}[z],$$

with the appropriate polynomials $B_0, \dots, B_m \in \mathbb{C}[z]$. Integrating by parts the right hand side, we obtain the required identity,

$$B_0(f)\,\omega = dg + h\,df, \qquad g = \sum_{j=1}^{m} B_j(f) g_j, \quad h = -\sum_{j=1}^{m} g_j \frac{dB_j}{dz}(f).$$

Apriori, the roots of B_0 may be arbitrary. We will show now that all of them except for those from the Bonnet set, can be eliminated by an appropriate division. Indeed, assume that $z \notin \mathrm{Bs}(f)$ and consider the representation

$$(f-z)\alpha = h\,df + dg, \qquad h, g \in \mathbb{C}[x,y] \tag{26.22}$$

for an arbitrary polynomial form $\alpha \in \Lambda^1[x,y]$. This representation implies that the polynomial g is locally constant along L_z, as its differential vanishes on the tangent vector to L_z at any smooth point of the latter. Since $z \notin \mathrm{Bs}(f)$, L_z is connected and therefore g is globally constant on L_z; without loss of generality we may assume that $g|_{L_z} = 0$. But the primary decomposition of $f - z$ in $\mathbb{C}[x,y]$ contains no multiple factors (otherwise L_z would carry nonisolated critical points of f). Therefore vanishing of g on L_z implies that g is divisible by $f - z$, $g = (f-z) \cdot g'$. Substituting this into (26.22), we conclude that

$$(f-z)\alpha = (f-z)dg' + [h + g']\,df. \tag{26.23}$$

Since df is nonvanishing at all noncritical points of L_z, the coefficient $h + g'$ must vanish on L_z and, hence, as before, it must be divisible by $f - z$: $h + g' = (f-z)h'$. Substituting it into (26.23), we see that the factor $f - z$ can be cancelled out from all three terms of it, yielding a new representation $\alpha = dg' + h'\,df$ with $g', h' \in \mathbb{C}[x,y]$.

Applying inductively this division procedure to the form $\alpha = B_0(f)\,\omega$, we eliminate all roots of the polynomial B_0 except for those that belong to the Bonnet set. At the end we arrive at the representation (26.20) with the product $b(x,y) = \prod_{\zeta_k \in \mathrm{Bs}(f)} (f(x,y) - \zeta_k)^{m_k}$ in the left hand side. □

26$\mathbf{D}_4$. *Polynomials transversal to infinity.* The condition of connectedness of affine level curves $L_z = f^{-1}(z)$ is intimately related to the behavior of the polynomial f "at infinity". There is a simple sufficient condition on the principal homogeneous terms of f, guaranteeing certain regularity of f at infinity, in particular, entailing that all curves L_z are connected. This condition will repeatedly appear in the future.

Definition 26.16. A polynomial $f \in \mathbb{C}[x, y]$ of degree $n + 1 \geqslant 2$ is called *transversal to infinity*, if one of the two equivalent conditions holds:

(1) its principal homogeneous part factors out as the product of $n + 1$ pairwise different linear forms;
(2) its principal homogeneous part has an isolated critical point of multiplicity n^2 at the origin.

The term "transversality" is explained by the following proposition, which is proved by an elementary computation in the affine chart covering the infinite line.

Proposition 26.17. *If f satisfies any of the two above conditions, then the projective compactification $\overline{L_z}$ of any level curve $L_z = f^{-1}(z)$ intersects transversally the infinite line $\mathbb{I} \subset \mathbb{P}^2$.* □

Corollary 26.18. *If f is transversal to infinity, then all affine level curves $L_z = f^{-1}(z) \subset \mathbb{C}^2$ are connected.*

Proof. Consider the irreducible decomposition of $\overline{L_z} = \bigsqcup_j C_j$. Any irreducible component $C_j \in \mathbb{P}^2$ is always connected, and any two irreducible components in $\mathbb{P}^2$ necessarily intersect (by the number of points equal to the product of their degrees, if counted with multiplicities; see [**Mum76**, §5B]). The intersection points of different components are necessarily singular and hence cannot lie on the infinite line by Proposition 26.17. Thus any two components intersect somewhere at the finite (affine) part $\mathbb{C}^2 \subset \mathbb{P}^2$, which means that the affine level curves curves are all connected. □

26E. Brieskorn lattice and Petrov modules. Theorem 26.13 allows us to describe algebraically the space of Abelian integrals as multivalued functions. They constitute a module over the ring $\mathbb{C}[z]$. The basis of this module will be computed in this section.

Let $f \in \mathbb{C}[x, y]$ be a polynomial.

Definition 26.19. The *Brieskorn lattice* is the quotient space

$$\mathbf{B}_f = \frac{\Lambda^2}{df \wedge d\Lambda^0}, \qquad \Lambda^{1,2} = \Lambda^{1,2}[x, y]. \tag{26.24}$$

The *Petrov module* is the quotient space

$$\mathbf{P}_f = \frac{\Lambda^1}{df \cdot \Lambda^0 + d\Lambda^0}, \qquad \Lambda^0 \cong \mathbb{C}[x,y] \tag{26.25}$$

of all polynomial 1-forms modulo the subspace of algebraically relatively exact forms. In the assumptions of Theorem 26.13, the Petrov module can be identified with the space of Abelian integrals.

Both $\mathbf{B}_f$ and $\mathbf{P}_f$ can be considered as $\mathbb{C}[f]$-modules: the generator f of the ring $\mathbb{C}[f]$ acts on equivalence classes of forms as the multiplication by the polynomial $f(x,y) \in \mathbb{C}[x,y]$. Definition of this action is correct, since

$$f\,df \wedge dg = df \wedge d(fg), \qquad f \cdot (h\,df + dg) = (fh - g)\,df + d(fg).$$

The exterior derivative d is a linear *bijection* $d\colon \mathbf{P}_f \to \mathbf{B}_f$ but *not* a $\mathbb{C}[f]$-module homomorphism.

In this subsection we explicitly construct the bases for these modules (actually, we will be mostly interested in $\mathbf{P}_f$)for polynomials f transversal to infinity.

Let f be a polynomial of degree $n+1$ transversal to infinity, with the principal homogeneous part f_{n+1}. The quotient space

$$Q_{df_{n+1}} = \frac{\Lambda^2}{df_{n+1} \wedge \Lambda^1} \cong \frac{\mathbb{C}[x,y]}{\left\langle \frac{\partial f_{n+1}}{\partial x}, \frac{\partial f}{\partial y} \right\rangle} \tag{26.26}$$

is a finite-dimensional complex algebra (cf. with Definition 8.22).

The (complex) dimension of the quotient (26.26) is equal to n^2. Indeed, both partial derivatives $\partial f_{n+1}/\partial x$ and $\partial f_{n+1}/\partial y$ are homogeneous polynomials and factor as products of exactly n *linear* forms each. No linear factor of $\partial f/\partial x$ can occur in $\partial f/\partial y$, otherwise the singularity of f_{n+1} will be non-isolated. By Proposition 8.25 the dimension of the quotient algebra is n^2.

Let $\omega_1, \dots, \omega_m$, $m = n^2$, be any homogeneous monomial 1-forms whose differentials $d\omega_1, \dots, d\omega_m$ generate the basis of $Q_{df_{n+1}}$. We will show that these differentials also generate the full quotient algebra $Q_{df} = \Lambda^2/df \wedge \Lambda^1$ over $\mathbb{C}$ and the Brieskorn lattice $\mathbf{B}_f$ over $\mathbb{C}[f]$, while the forms themselves generate the Petrov module $\mathbf{P}_f$. This result has several different demonstrations (see, e.g., [**KP06**] which treats the multidimensional case as well). We follow the exposition in [**Yak02**] which has an advantage of being purely algebraic and effective.

Proposition 26.20. *Assume that the polynomial $f = f_{n+1} + \cdots$ is transversal to infinity.*

Then any collection of 2-forms $d\omega_j$, $j = 1, \dots, n^2$, which generates the quotient $Q_{df_{n+1}} = \Lambda^2/df_{n+1} \wedge \Lambda^1$ over $\mathbb{C}$, generates also the quotient $Q_{df} = \Lambda^2/df \wedge \Lambda^1$.

Proof. By the choice of the forms $d\omega_j$, any 2-form $\mu \in \Lambda^2$ can be "divided with remainder" by df_{n+1} as follows:

$$\mu = \sum_1^m c_j\, d\omega_j + df_{n+1} \wedge \eta, \qquad \eta \in \Lambda^1$$

with the "incomplete ratio" η. Substitute in this equality $f_{n+1} = f - \varphi$, where φ is the collection of all nonprincipal terms of f, $\deg \varphi \leqslant n$. Then $\mu = \sum_1^m c_j\, d\omega_j + df \wedge \eta - \mu'$, where $\mu' = \varphi \wedge \eta$ is a 2-form of degree *strictly smaller* than $\deg \mu$. The division process can therefore be continued inductively until the "incomplete ratio" disappears. □

Theorem 26.21. *Let $f = f_{n+1} + \cdots \in \mathbb{C}[x, y]$ be a polynomial of degree $n+1$ transversal to infinity and $d\omega_1, \dots, d\omega_m$, a collection of $m = n^2$ monomial 2-forms generating the quotient algebra $Q_{df_{n+1}} = \Lambda^2/df_{n+1} \wedge \Lambda^1$.*

Then any 1-form $\omega \in \Lambda^1$ can be represented as follows:

$$\omega = \sum_{j=1}^m (p_j \circ f)\, \omega_j + h\, df + dg \tag{26.27}$$

with some polynomials $g, h \in \mathbb{C}[x, y]$ and the univariate polynomials $p_1, \dots, p_m \in \mathbb{C}[z]$. The degrees of the coefficients p_j satisfy the equalities

$$(n+1) \deg_z p_j + \deg \omega_j = \deg \omega, \qquad j = 1, \dots, m. \tag{26.28}$$

Corollary 26.22. *In the assumptions of the Theorem 26.21, the forms $\omega_1, \dots, \omega_m$ generate the Petrov module $\mathbf{P}_f$ over $\mathbb{C}[f]$.* □

We begin the proof with the following lemma which may be considered as an analog of the Euler identity in the Brieskorn lattice. Let f_{n+1} be an arbitrary *homogeneous* polynomial of degree $n + 1$.

Lemma 26.23. *Any polynomial 2-form divisible by df_{n+1}, has a primitive divisible by f_{n+1} in the Brieskorn lattice $\mathbf{B}_{f_{n+1}}$.*

In other words, for any 1-form $\eta \in \Lambda^1$ there exists $\omega \in \Lambda^1$ and such that

$$df_{n+1} \wedge \eta = d(f_{n+1}\, \omega) \mod df_{n+1} \wedge d\Lambda^0. \tag{26.29}$$

Proof of the lemma. By the Euler identity, we have $f_{n+1} = \frac{1}{n+1} i_V\, df_{n+1}$, where $V = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}$ is the Euler field and $i_V \colon \Lambda^k \to \Lambda^{k-1}$ the antiderivation substituting V as the first argument of a differential k-form.

Since any 3-form on the 2-plane vanishes, $df_{n+1} \wedge \mu = 0$ for any 2-form μ. Applying the antiderivation i_V, we conclude from this and the Euler formula above, that for any 2-form μ,

$$\begin{aligned} 0 &= i_V(df_{n+1} \wedge \mu) = (i_V\, df_{n+1}) \wedge \mu - df_{n+1} \wedge (i_V \mu) \\ &= (n+1) f_{n+1} \mu - df_{n+1} \wedge (i_V \mu). \end{aligned} \tag{26.30}$$

Using (26.30), the equation (26.29) with respect to the unknown 1-form ω can be transformed as follows,

$$df_{n+1} \wedge \eta = df_{n+1} \wedge \omega + \tfrac{1}{n+1}\, df_{n+1} \wedge (i_V\, d\omega) + df_{n+1} \wedge d\xi.$$

The latter is an equation to be solved now with respect to ω and ξ. It will be obviously satisfied if

$$\eta = \tfrac{1}{n+1}\, i_V\, d\omega + \omega + d\xi,$$

or, after applying the derivation d to both sides,

$$d\eta = \tfrac{1}{n+1}\, d(i_V \mu) + \mu, \qquad \mu = d\omega,$$

(the derivation results in an equivalent condition since any closed polynomial form on $\mathbb{C}^2$ is exact).

To show that the last equation is always solvable with respect to μ for any 2-form $d\eta$, we transform it for the last time using the homotopy formula $L_V = di_V + i_V d$ [**Arn97**] and the fact that $d\mu = 0$ (as a 3-form on the 2-plane). Finally the equation (26.29) is reduced to the equation

$$d\eta = \big(\tfrac{1}{n+1}\, L_V + \mathrm{id}\big)\mu, \qquad \mu = d\omega, \tag{26.31}$$

where L_V is the Lie derivative acting on 2-forms. Since V is the Euler field, each monomial 2-form $x^p y^q\, dx \wedge dy$ is an eigenvector for L_V with the positive eigenvalue $p + q + 2$. Thus $\frac{1}{n+1} L_V + \mathrm{id}$ is a diagonalizable operator with positive eigenvalues on the space of polynomial forms of any degree. Such an operator is invertible, which yields a solution to (26.31) and ultimately to (26.29). □

Remark 26.24. A similar argument shows that a 2-form divisible by df_{n+1} is also divisible by f_{n+1} in the Brieskorn lattice (i.e., modulo $df_{n+1} \wedge d\Lambda^0$); see [**Yak02**].

Proof of Theorem 26.21. The proof imitates demonstration of Proposition 26.20. By assumption, the forms $d\omega_1, \dots, d\omega_m$ form a basis of the quotient algebra $Q_{df_{n+1}}$ associated with the principal homogeneous part f_{n+1} of the polynomial f. This means that the 2-form $d\omega \in \Lambda^2[\mathbb{C}^2]$ can be uniquely represented as

$$d\omega = \sum_{j=1}^{m} c_j\, d\omega_j + df_{n+1} \wedge \eta, \tag{26.32}$$

where $\eta \in \Lambda^1[\mathbb{C}^2]$ is a polynomial 1-form.

By Lemma 26.23, the "incomplete ratio" $df_{n+1} \wedge \eta$ can be rewritten as $d(f_{n+1}\omega') + df_{n+1} \wedge dg$ for some polynomial $g \in \Lambda^0$. Passing to the primitives, we conclude that

$$\omega - \sum_{j=1}^{m} c_j\omega_j = f_{n+1}\omega' - g\,df_{n+1} + dh.$$

Substitute f_{n+1} by $f - \varphi$, where $\varphi \in \mathbb{C}[x, y]$ is the collection of all nonprincipal terms of f, $\deg\varphi \leqslant n$. After collecting terms, we obtain the equality $\omega - \sum_{j=1}^{m} c_j\omega_j = f\omega' - g\,df + dh - \omega''$, $\omega'' = \varphi\omega' - g\,d\varphi$. In other words, we have

$$\omega - \sum c_j\omega_j = f\omega' - \omega''$$

in the $\mathbb{C}[f]$-module $\mathbf{P}_f$; cf. with (26.25). The degrees of both 1-forms ω', ω'' are strictly less than that of ω, therefore the process can be continued by induction, resulting at the end in the representation (26.28).

The assertion on the degrees follows directly from inspection of this division algorithm. □

Remark 26.25. A similar argument shows that any 2-form $\mu \in \Lambda^2[\mathbb{C}^2]$ can be represented as the sum $\sum_{j=1}^{m}(q_j \circ f)\,d\omega_j \bmod df \wedge d\Lambda^0$. Moreover, both assertions admit natural generalizations for polynomials $f \in \mathbb{C}[x_1, \dots, x_k]$, $k \geqslant 2$, whose principal *quasi*homogeneous part has an isolated critical point at the origin. Details can be found in [**Yak02**].

26F. Polynomials as topological bundles. In this section we study the analytic continuation of Abelian integrals as multivalued functions of one complex variable. The ramification locus of any Abelian integral and its monodromy are completely determined by the Hamiltonian f. In what follows we denote by Σ the set of *critical values* of f,

$$\Sigma = \bigl\{z \in \mathbb{C}\colon \exists(x, y) \in \mathbb{C}^2,\ \tfrac{\partial f}{\partial x} = \tfrac{\partial f}{\partial y} = f - z = 0 \text{ at } (x, y)\bigr\}. \qquad (26.33)$$

Theorem 26.26. *If a polynomial $f \in \mathbb{C}[x, y]$ is transversal to infinity, then the map $f\colon \mathbb{C}^2 \to \mathbb{C}$ defines a topological bundle over the set of all noncritical values $\mathbb{C} \smallsetminus \Sigma$.*

In other words, for any $a \in \mathbb{C} \smallsetminus \Sigma$ there exists a small neighborhood $U \ni a$ in $\mathbb{C} \smallsetminus \Sigma$ such that the full preimage $f^{-1}(U)$ is homeomorphic to the Cartesian product $f^{-1}(a) \times U$ and f restricted on this preimage is topologically conjugate by this homeomorphism to the projection of $f^{-1}(a) \times U$ on the second term.

This result obviously follows from the general Theorem 26.27 below. Consider the complex *affine* space P of all bivariate polynomials of degree $n + 1$ with the fixed principal square-free homogeneous part f_{n+1} (all such polynomials are by definition transversal to infinity). The dimension of this space is $r = (n+2)(n+3)/2$, $P \cong \mathbb{C}^r$, and it can be identified with the space of nonprincipal coefficients λ_{ij} in the expansion

$$\varPhi(\lambda; x, y) = f_{n+1}(x, y) + \sum_{0 \leqslant i+j \leqslant n} \lambda_{ij} x^i y^j. \tag{26.34}$$

In the product space $P \times \mathbb{C}^2$ fibered over P consider the algebraic hypersurface $X = \{\varPhi(\lambda, x, y) = 0\}$ and its fiberwise compactification $\overline{X} \subset P \times \mathbb{P}^2$. Denote by $\pi \colon \overline{X} \to P$ the natural projection $P \times \mathbb{P}^2 \to P$ restricted on the surface $\overline{X}$. The preimages $\pi^{-1}(\lambda) \subset \{\lambda\} \times \mathbb{P}^2$ are projectively compactified zero level curves of the polynomial $\varPhi_\lambda(x, y) = \varPhi(\lambda, x, y) \in \mathbb{C}[x, y]$. The preimages by the projection $\pi \colon \overline{X} \to P$ run over all level curves of all polynomials with the fixed principal homogeneous part.

Let $\boldsymbol{\Sigma} \subset P$ be the set of all parameters λ for which the affine curve $\{\varPhi_\lambda(x, y) = 0\} \subset \mathbb{C}^2$ is singular (nonsmooth). By definition (compare with (26.33)),

$$\boldsymbol{\Sigma} = \bigl\{\lambda \in P \colon \exists (x, y) \in \mathbb{C}^2 \colon \tfrac{\partial \varPhi}{\partial x} = \tfrac{\partial \varPhi}{\partial y} = \varPhi = 0 \text{ at } (\lambda, x, y)\bigr\}. \tag{26.35}$$

Theorem 26.27. *If the principal homogeneous part f_{n+1} is square-free, then the projection $\pi \colon \overline{X} \to P$ and its restriction on the affine part $X \subset \overline{X}$ are topologically locally trivial bundles over the complement $P \smallsetminus \boldsymbol{\Sigma}$.*

Proof. 1. For any point $a = (\lambda, x, y) \in \overline{X}$ over $\lambda \notin \boldsymbol{\Sigma}$ the complex tangent space $\mathbf{T}_a X$ at this point projects surjectively onto the tangent space $\mathbf{T}_\lambda P \cong P \cong \mathbb{C}^r$ at the point $\lambda = \pi(a)$.

For points in the affine part $X \subset \overline{X}$ this follows from the fact that one of the partial derivatives $\partial\varPhi/\partial x$ or $\partial\varPhi/\partial y$ is nonvanishing at $a \in X$ by the assumption $\lambda \notin \boldsymbol{\Sigma}$.

For points at infinity the above surjectivity assertion holds regardless of the choice of λ, since in the suitable coordinates $v = 1/x$, $u = y/x$ the equation of $\overline{X}$ takes the form $\{\Psi = 0\}$, where $\Psi(\lambda; u, v) = f_{n+1}(1, u) + v g_1(\lambda; u, v)$. Since $f_{n+1}(1, u)$ has $n+1$ distinct *simple* roots, the derivative $\partial\Psi/\partial u$ does not vanish on the infinite line $v = 0$ for all λ.

2. Let $F_1, \dots, F_{2r}$ be commuting vector fields on the base P, spanning the tangent bundle $\mathbf{T}P$ (e.g., the fields $\partial/\partial\lambda_{ij}$ and $\sqrt{-1}\,\partial/\partial\lambda_{ij}$ for all i, j). The above surjectivity means that the preimage $\pi^{-1}(U) \subset U \times \mathbb{P}^2$, where $U \subset P \smallsetminus \boldsymbol{\Sigma}$ is a sufficiently small open set disjoint with $\boldsymbol{\Sigma}$, can be covered by a union of open sets $U_\alpha \subset \overline{X}$ such that in each neighborhood there exist $2r$ real analytic vector fields $\overline{F}_{k,\alpha}$ tangent to $\overline{X}$ and π-related with F_k for all k; see Fig. V.2. Moreover, the fields $\overline{F}_{k,\alpha}$ can be assumed tangent to the intersection of $\overline{X}$ with the infinite line in each fiber. Indeed, since $\partial\Psi/\partial u \neq 0$, the v-component of such a lift (in the chart $v = 1/x$, $u = y/x$ as above) can be chosen arbitrarily, in particular, zero.

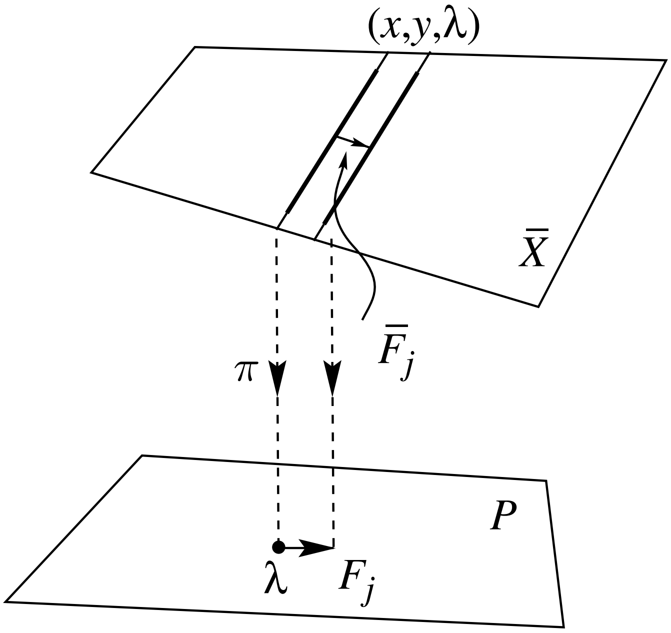

Figure V.2. Topological trivialization of the map

3. Since π is proper (all preimages of points are compact projective curves), one may assume that the covering is finite. Let $\{\psi_\alpha \geqslant 0\}$ be a partition of unity subordinated to the covering U_α. Then the vector fields $\overline{F}_k = \sum_\alpha \psi_\alpha \overline{F}_{k,\alpha}$ are also π-related with F_k and tangent to $\overline{X}$. Since F_k commute, the commutators $[\overline{F}_k, \overline{F}_j]$ are tangent to the fibers $\pi^{-1}(\lambda)$. By a standard modification, one can make the fields $\overline{F}_k$ also commuting in $\mathbf{T}\overline{X}$ by adding appropriate vector fields tangent to the fibers; see [**War83**].

4. Shifts along the commuting vector fields $\overline{F}_k$ realize homeomorphisms between all fibers $\pi^{-1}(\lambda)$, $\lambda \in U$, and trivialize the map $\pi \colon \overline{X} \to P$. □

Corollary 26.28. *In the assumptions of Theorem 26.26, any cycle $\delta(z_*) \in H_1(L_{z_*}, \mathbb{Z})$ can be continued along any path $\gamma\colon [0,1] \to \mathbb{C} \smallsetminus \Sigma$, $\gamma(0) = z_*$, avoiding critical values of f. The result is defined uniquely as a cycle in the homology group and does not change when γ is replaced by a homotopy equivalent path.* □

This corollary allows us to consider the effect of continuous deformations of affine level curves L_z as z goes along closed loops in the z-plane avoiding the critical set Σ. With any such closed loop γ (beginning and ending at a certain fixed regular base point $z_* \notin \Sigma$), the *topological monodromy operator*

$$\Delta_\gamma \colon H_1(L_{z_*}, \mathbb{Z}) \to H_1(L_{z_*}, \mathbb{Z}), \tag{26.36}$$

can be associated: together these operators constitute the representation of the fundamental group $\pi_1(\mathbb{C} \smallsetminus \Sigma, t_*)$ (as usual, the choice of the base point is not important). We will show that in general the operators Δ_γ are *nontrivial*: after continuation along closed paths the cycles on the level curves in general do not return to their initial positions.

We conclude this analysis with the following basic result.

Theorem 26.29 (analytic continuation of Abelian integrals)**.** *If f is a polynomial transversal to infinity, then for any polynomial 1-form ω and any cycle $\delta \in H_1(L_{z_*}, \mathbb{Z})$, the Abelian integral $\oint_\delta \omega$ can be extended as an analytic multivalued function along any path avoiding the critical values of f.*

Proof. The restriction of any polynomial 1-form on L_z is closed (holomorphic), hence its integral is the same for all homotopic loops. By Theorem 26.26, for any fixed $z_* \notin \Sigma$ one can choose a representative of the cycle $\delta(z)$ continuously depending on z in a sufficiently small neighborhood of z_* in such a way that its projection, say, on the x-plane parallel to the y-axis is the same curve denoted by D. Then one can choose an analytic branch $y = y(x, z)$ of solution of the equation $f(x, y) = z$ over D. The integral of any form along $\delta(z)$ can be reduced to an integral over D of an analytic 1-form depending analytically on z. The rest follows from the standard theorem on (complex) differentiability of integrals depending on parameters. □

Remark 26.30. Without some assumptions "on infinity" the assertion of Theorem 26.26 and all its corollaries fails. In general there may exist regular values of f such that the preimages $f^{-1}(z)$ change their topological type at these points, exhibiting singularities on the infinite line $\mathbb{I}$. Such values are called *atypical values* of the polynomial f, but they also always form a finite subset of $\mathbb{C}$ (empty when f is transversal to infinity).

Theorem 26.31. *If f is a polynomial of degree $n+1$ transversal to infinity, then the rank of the first homology of any nonsingular fiber is equal to n^2.*

Proof. Consider first the case where the polynomial f is homogeneous and coincides with its principal part f_{n+1}. For such a polynomial the affine level curves are L_z are all affine equivalent to any one of them, say, to L_1. The surface L_1 is a compact Riemann surface of some genus g with $n+2$ deleted points. The genus g can be computed by the Riemann–Hurwitz formula [**For91**]. The projection $(x, y) \mapsto x$ restricted on L_1 defines the ramified covering $\overline{L_1} \to \mathbb{P}^1$ of multiplicity $m = n + 1$. The ramification points of the covering are defined by the equation $\frac{\partial f}{\partial y} = 0$ which is a homogeneous polynomial equation of degree n. The system of equations $f(x, y) = 1$, $\frac{\partial f}{\partial y}(x, y) = 0$ has therefore $n(n + 1)$ solutions, none of them at infinity and all simple. Indeed, the polynomial $\frac{\partial f}{\partial y}$ factors as the product of n linear terms corresponding to distinct lines on the (x, y)-plane. Restriction of the homogeneous polynomial f on each line is a homogeneous polynomial of degree $n+1$ in one variable (the local parameter along the line), which must be nonzero since in the opposite case f would have a multiple linear factor. All roots of the equation $ct^{n+1} = 1$, $c \neq 0$, are simple and distinct, hence each of n lines contributes exactly $n+1$ simple solutions to the system $f = 1$, $\frac{\partial f}{\partial y} = 0$.

Each simple solution of the system $f = 1$, $\frac{\partial f}{\partial y} = 0$ corresponds to a ramification point of index $k_j = 2$. By the Riemann–Hurwitz formula the total genus is

$$g = 1 - m + \sum_j \frac{k_j - 1}{2} = -n + \frac{1}{2} n(n+1) = \tfrac{1}{2} n(n-1). \tag{26.37}$$

The first homology group of each fiber L_z is generated by $2g$ *canonical loops* forming the basis of the homology of the projective compactification $\overline{L_z}$ and any n out of $n+1$ small loops around the deleted points at infinity (the sum of all $n+1$ small loops is homologous to zero). Any closed loop on L_z is homologous to a linear combination of these cycles with integral coefficients (since $H_1(\overline{L_z}, \mathbb{Z})$ is the free group generated by the canonical loops). Ultimately we have

$$\operatorname{rank} H_1(L_z, \mathbb{Z}) = 2g + n = n(n-1) + n = n^2.$$

This proves the assertion on the genus of curves when f is a homogeneous polynomial.

If f is not homogeneous and $f - f_{n+1} \neq 0$, the level curves $L_z = \{f = z\}$ for large values of z are perturbations of the level curves $\widetilde{L}_z = \{f_{n+1} = z\}$. Since the latter are nonsingular for $z \neq 0$, by the implicit function theorem L_z and $\widetilde{L}_z$ are diffeomorphic for sufficiently large values of z. But *all* curves L_z with $z \notin \Sigma$ are diffeomorphic to each other, therefore the rank of any homology group is the same *everywhere*, $\operatorname{rank} H_1(L_z, \mathbb{Z}) = \operatorname{rank} H_1(\widetilde{L}_1, \mathbb{Z}) = n^2$. □

26G. Gelfand–Leray derivative. The derivative of an Abelian integral is again an Abelian integral. More precisely, we have the following rule of derivation of Abelian integrals.

Theorem 26.32. *Let ω and η be two rational 1-forms such that*

$$d\omega = df \wedge \eta, \tag{26.38}$$

and $\delta(z) \in H_1(L_z, \mathbb{Z})$, $z \notin \Sigma$, a continuous family of cycles on noncritical level curves of f, not passing through poles of neither ω nor η. Then

$$\frac{d}{dz} \oint_{\delta(z)} \omega = \oint_{\delta(z)} \eta. \tag{26.39}$$

Proof. Consider the real 2-dimensional cylindrical surface M^2 in $\mathbb{C}^2$, parameterized by real parameters $s \in \mathbb{R} \bmod \mathbb{Z}$ and $t \in [0, \varepsilon]$ so that each circle $\{t = \text{const}\}$ parameterizes the cycle $\delta(z+t)$ on the level surface L_{z+t}. Such a surface exists for sufficiently small $\varepsilon > 0$, since the foliation $\{df = 0\}$ has trivial holonomy along the loop $\delta(z)$.

The boundary of M consists of two cycles, $-\delta(z)$ and $\delta(z+\varepsilon)$, hence by the Stokes theorem,

$$\oint_{\delta(z+\varepsilon)} \omega - \oint_{\delta(z)} \omega = \iint_M d\omega = \iint_M df \wedge \eta. \tag{26.40}$$

The form df vanishes on all cycles $\delta(z+t)$, so that the double integral in (26.40) reduces to the iterated integral

$$\int_0^\varepsilon dt \cdot \oint_{\delta(z+t)} \eta.$$

Dividing both parts of the equality by ε and passing to the limit as $\varepsilon \to 0$, we conclude with the formula (26.39): the convergence follows from the assumptions on the cycle $\delta(z)$. □

Example 26.33. Let $f(x,y) = x^2 + y^2$ and $\delta(z)$ for $z > 0$ be the real circle oriented counterclockwise. Then the Abelian integral $\oint_{\delta(z)} \omega$ of the 1-form $\omega = y\,dx$ is equal to $-\pi z$ (the area of the circle). The equation (26.38) is satisfied by the meromorphic 1-form $\eta = \frac{1}{2}\frac{dx}{y}$. This form has poles on the real cycles $\delta(z)$, but the restriction of η on all level curves L_z is holomorphic ($\eta|_{L_z}$ has removable singularity at the points with $y = 0$). Hence the cycles $\delta(z)$ can be moved off the polar locus of η without changing the integrals, while permitting application of Theorem 26.32. The integral of η along the cycles is identically equal to $-\pi$. This example allows us to recall the order of terms in the wedge product (26.38).

The Gelfand–Leray derivative of a polynomial 1-form is only *rational* on $\mathbb{C}^2$ and *nonunique*. However, its *restriction* on the nonsingular affine level curves $L_z = \{f = z\}$ is a uniquely defined holomorphic 1-form from $\Lambda^1(L_z)$. Indeed, the derivative is defined uniquely modulo a rational 1-form on $\mathbb{C}^2$ having zero restriction on the fibers (the only solutions of the equation $\eta \wedge df = 0$). On the other hand, locally near a point $a \in L_z$ at which $\frac{\partial f}{\partial y} \neq 0$, the Gelfand–Leray derivative of a form ω with $d\omega = A(x,y)\,dx \wedge dy$ can be obtained by restriction on L_z of the holomorphic form $\eta = -\frac{A(x,y)}{\frac{\partial f}{\partial y}(x,y)}\,dx$.

The Gelfand–Leray derivative is often denoted by $\frac{d\omega}{df}$: while this notation is ambiguous if used for a rational form in $\mathbb{C}^2$, the ambiguity disappears after restriction on the level curves.

Remark 26.34. If $d\omega = A(x,y)\,dx \wedge dy$ has a polynomial coefficient $A \in \mathbb{C}[x,y]$ of degree m and f is of degree $n+1$ transversal to infinity, then the Gelfand–Leray derivative $\frac{d\omega}{df}$ restricted on the level curves L_z has a pole of order $\leqslant m-n+2$ at each of the $n+1$ points at infinity. Indeed, the partial derivative $\frac{\partial f}{\partial y}$ restricted on each analytic branch of the curve $\{f=z\}$, has a pole of order exactly n and no smaller. This follows from computations with principal homogeneous terms: arguments similar to those proving Theorem 26.31, show that the leading coefficient of the partial derivative cannot vanish. Thus the derivative η has the form $O(x^{m-n})\,dx$, at infinity, i.e., the pole of order $\leqslant m-n+2$ in the local chart $u=1/x$.

In particular, if $\omega = P\,dx + Q\,dy$ with $\deg P, Q \leqslant n$, and f is transversal to infinity of degree $n+1$, then the order of the pole of $\frac{d\omega}{df}$ is at most 1, i.e., all poles at infinity are simple.

26H. Picard–Fuchs system and its properties. Consider a polynomial $f \in \mathbb{C}[x,y]$ of degree $n+1$, transversal to infinity, and let $\boldsymbol{\omega} = (\omega_1, \dots, \omega_m)$, $m = n^2$, be the tuple of monomial 1-forms generating the Petrov module $\mathbf{P}_f$ as in Theorem 26.21.

Theorem 26.35. *For any continuous family of cycles $\delta(z)$ on the level curves, the column vector $X = X(z)$ of the periods $\oint_{\delta(z)} \omega_j$, $j = 1, \dots, m$, satisfies the following system of linear ordinary differential equations,*

$$(zE - A) \cdot \frac{dX}{dz} = (B_0 + zB_1) \cdot X. \tag{26.41}$$

Here A, B_0, B_1 are constant $m \times m$-matrices.

Proof. Consider the 2-forms $f\,d\omega_j \in \Lambda^2$, for all $j = 1, \dots, m$. Each of them can be "divided with remainder" by df: by Proposition 26.20 there exist polynomial 1-forms η_j of degrees $\deg \eta_j \leqslant \deg \omega_j$ such that

$$f\,d\omega_j = df \wedge \eta_j + \sum_{k=1}^{m} a_{jk}\,d\omega_k \tag{26.42}$$

with some complex numbers $a_{jk} \in \mathbb{C}$ forming an $m \times m$-matrix A. These identities can be rewritten under the form

$$d(f\omega_j - \sum_k a_{jk}\omega_k) = df \wedge (\eta_j + \omega_j). \tag{26.43}$$

The 1-forms η_j can be in turn expanded using Theorem 26.21, as

$$\eta_j = \sum_{k=1}^{m} (b_{jk} \circ f) \cdot \omega_k \mod d\Lambda^0 + \Lambda^0\, df, \tag{26.44}$$

with some univariate polynomials $b_{jk}(z)$ which have to be composed with f. Since $\deg \eta_j \leqslant 2n < 2(n+1)$, the degrees of these polynomials by (26.28) cannot exceed 1. Together these polynomials can be arranged into a linear matrix polynomial $B(z) = \|b_{jk}(z)\|_{j,k=1}^{m} = B_0 + zB_1$.

Note that the f restricted on $\delta(z)$ is identically equal to z. Therefore applying Theorem 26.32, we conclude that

$$\frac{d}{dz}(zX(z) - A \cdot X(z)) = (E + B_0 + zB_1) \cdot X(z). \tag{26.45}$$

Since A is constant, the relationship (26.45) is equivalent to (26.41). □

Many properties of the matrices A, B_0, B_1 can be seen from their explicit construction in the proof of Theorem 26.35.

Corollary 26.36. *If f has only nondegenerate critical points, then A is a diagonalizable matrix whose spectrum consists of the corresponding critical values of f.*

Proof. By construction, A is the matrix of multiplication by f in the quotient algebra $\Lambda^2/df \wedge \Lambda^1 \cong Q_{df}$. If f has only nondegenerate critical points (as usual, being transversal to infinity), then the quotient algebra is isomorphic to the algebra of functions on m distinct points forming the critical locus of f in $\mathbb{C}^2$ and the spectrum of A consists of the critical values of f, counted with their multiplicities if some critical values at distinct critical points coincide. □

This in turn implies the following conclusion.

Corollary 26.37. *If f is a Morse polynomial transversal to infinity, then the linear system* (26.41) *has only Fuchsian singularities at the critical values of the polynomial f.*

Proof. In the assumptions of the corollary, the determinant $\det(zE - A)$ has only simple roots at the critical values of f, hence the inverse matrix $(zE - A)^{-1}$ has simple poles there. □

Remark 26.38. In a similar way the matrix coefficients of the decomposition B_0, B_1 can be described. In particular, their norms can be, if necessary, estimated from above in terms of the relative magnitude of principal and nonprincipal homogeneous coefficients of f.

Remark 26.39. The singular point of the system (26.41) at infinity is in general non-Fuchsian (though obviously always regular); see [**Nov02**]. However, if instead of $m = n^2$ forms $d\omega_j$ generating the algebra $\mathbf{Q}_f$ we take all $\nu = n(2n-1)$ monomial 2-forms of degree $\leqslant 2n$, then in the first division (26.42) one can achieve $\deg \eta_j \leqslant \deg \omega_j \leqslant 2n$ and hence one can always represent $d\eta_j$ as a linear combination of the forms $d\omega_k$ with *constant complex* coefficients. This will produce a *redundant* system of linear differential equations satisfied by the vector of periods $X'(z)$ of all ν 1-forms in the *hypergeometric form*,

$$(zE - A')\frac{d}{dz}X'(z) = B'X'(z), \qquad A', B' \in \operatorname{Mat}(\nu, \mathbb{C}). \tag{26.46}$$

Such a system always has a Fuchsian singular point at infinity. For details see [**NY01**].

26I. Vanishing cycles and Picard–Lefschetz formulas. In this section we compute the monodromy operators (26.36) associated with a particular case where the loop γ is a small path encircling just one Morse critical value of the polynomial f.

As z tends to a nondegenerate critical value $a \in \Sigma$ of f, among all cycles on the curve L_z one can distinguish a certain cycle $\delta_a(z) \in H_1(L_z, \mathbb{Z})$ called the *vanishing cycle*. This cycle is defined uniquely modulo orientation (i.e., up to multiplication by -1 in the group $H_1(L_z, \mathbb{Z})$).

To describe the vanishing cycle accurately, assume that the critical value of the polynomial f is $a = 0$ and the corresponding nondegenerate critical point is at the origin. Without loss of generality we may further assume that $f(x, y) = x^2 - y^2 + \cdots$. All this can be achieved by affine changes of the variables x, y and z.

Consider the parallel projection on the x-plane parallel to the y-axis, restricted on different level curves L_z. These restrictions have critical points depending on z, when the derivative $\frac{\partial f}{\partial y}$ vanishes on L_z: near all other points the curve L_z locally biholomorphically covers the x-plane.

Out of those critical points, there are exactly two points near the origin, defined by the equations

$$x^2 - y^2 + \cdots = z, \qquad 2y + \cdots = 0$$

(as usual, the dots denote higher order terms). Resolving these equations, we conclude that L_z can be locally described as the Riemann surface covering the x-plane $(\mathbb{C}, 0)$ with ramification at the two near the origin, $x_\pm(z) = \pm\sqrt{z + \cdots}$, the ramification having order 2. The loop on the x-plane, which encircles these points, can be lifted to a cycle $\delta_0(z)$ on the level curve L_z for all $z \neq 0$ sufficiently close to zero; see Fig. V.3.

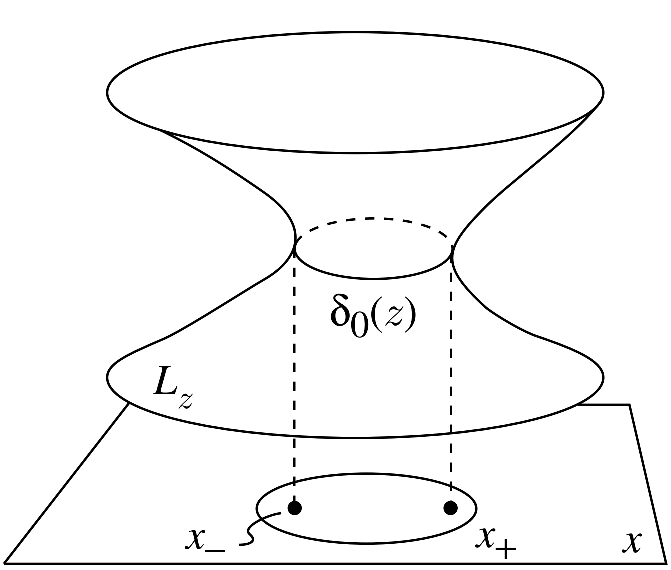

Figure V.3. Vanishing cycle at a Morse singularity

Definition 26.40. The cycle $\delta_0(z) \in H_1(L_z, \mathbb{Z})$ defined by this construction for all $z \neq 0$ sufficiently close to the critical value $z_0 = 0$, is called the *vanishing cycle* (more precisely, the cycle vanishing at the critical value z_0).

Remark 26.41. The vanishing cycle (modulo orientation and the free homotopy deformation on L_z) can be characterized by the following purely topological property: *as $z \to 0$, the cycle $\delta_0(z)$ can be represented by a continuous family of loops on L_z of length that tends to zero.* This description explains the terminology.

Now we can describe the monodromy operator for a small loop encircling a Morse critical value. Suppose that the regular value z varies along the small circular loop γ around the origin, $z(t) = \rho e^{2\pi i t}$, $0 < \rho \ll 1$, parameterized by the real variable $t \in [0, 1]$. Then the two points $x_\pm(z(t))$ also rotate along two curves approximating two half-circles $x_\pm(t) = \pm\sqrt{\rho}e^{\pi i t}(1 + o(1))$ and at the end exchange their places; see Fig. V.4.

Looking at this figure, one can construct a continuous isotopy of the plane, which is identical outside the disk of radius, say, $3\sqrt{\rho}$ and a rotation by π on the disk of radius $2\sqrt{\rho}$. This isotopy of the plane lifts as an isotopy of the fiber L_ρ on itself, identical outside a small disk centered at the critical point, called the *Dehn twist*.

The action of the Dehn twist on the vanishing cycle $\delta_0(\rho)$ itself is trivial: the cycle "rotates" along itself. However, if $\delta'(\rho)$ is another cycle which intersects the vanishing cycle, then on the level of homology the Dehn twist

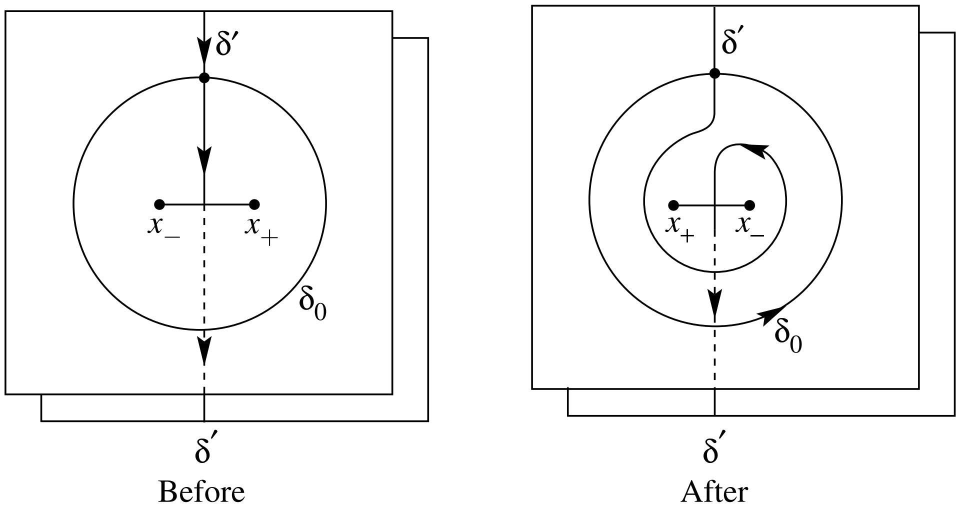

Figure V.4. Vanishing cycles and monodromy around a Morse singularity

acts by adding to $\delta'(\rho)$ the vanishing cycle $\delta_0(\rho)$ with the sign $\pm$ depending on the intersection index $\delta_0 \cdot \delta'$ between δ_0 and δ'. If δ' is a simple curve, this is instantly clear from Fig. V.4; for cycles having multiple intersections with δ_0 one has to use the additivity of the intersection index to prove the following result, called the *Picard–Lefschetz formula*:

$$\Delta_\gamma \delta = \delta + (\delta \cdot \delta_0)\, \delta_0. \tag{26.47}$$

Remark 26.42. The formal construction of the Dehn twist and the proof of Picard–Lefschetz formulas can be found in numerous sources, among them [**AGV88, Pha67, DFN85**]. In the local context, where only the intersection of the level curves with a small ball around the critical point are considered, one should exercise a certain care distinguishing between the *absolute* and *relative* (modulo the boundary) homology; see [**AGV88**, §1].

As an immediate corollary from the Picard–Lefschetz formulas, we obtain the following. Choose any basis $\boldsymbol{\delta} = \{\delta_1, \dots, \delta_m\}$ in the homology $H_1(L_a, \mathbb{Z})$ for some regular value $a \in \mathbb{C} \smallsetminus \Sigma$, considered as a row vector. Then analytic continuation Δ_γ along any loop $\gamma \in \pi_1(\mathbb{C} \smallsetminus \Sigma, a)$ is represented by a matrix M_γ as follows:

$$\Delta_\gamma \boldsymbol{\delta} = \boldsymbol{\delta} \cdot M_\gamma, \qquad M_\gamma \in \operatorname{Mat}(m, \mathbb{Z}). \tag{26.48}$$

Proposition 26.43. *All monodromy matrices M_γ defined above, are unimodular,* $\det M_\gamma = 1$.

Proof. If γ is a small loop around a Morse critical value, then the equality $\det M_\gamma = 1$ follows immediately from (26.47) (the vanishing cycle δ_0 can always be chosen as the first element in the row $\boldsymbol{\delta}$).

An arbitrary loop is a product of several small loops as above. □

26J. Period matrices. Let f be a polynomial of degree $n+1$ transversal to infinity, and $\boldsymbol{\omega} = (\omega_1, \dots, \omega_m)$, $m = n^2$, an arbitrary tuple of polynomial 1-forms. With this tuple, considered as a column vector, and any choice of a locally constant basis $\boldsymbol{\delta}(z) = (\delta_1(z), \dots, \delta_m(z))$ in the homology of the level curves L_z, $z \in \mathbb{C} \smallsetminus \Sigma$, one can associate the *period matrix*

$$X(z) = \boldsymbol{\omega} \otimes \boldsymbol{\delta}(z) = \begin{pmatrix} \oint_{\delta_1(z)} \omega_1 & \cdots & \oint_{\delta_m(z)} \omega_1 \\ \vdots & \ddots & \vdots \\ \oint_{\delta_1(z)} \omega_m & \cdots & \oint_{\delta_m(z)} \omega_m \end{pmatrix} \tag{26.49}$$

which is an analytic multivalued matrix-function ramified over the locus Σ.

As follows from (26.48), the analytic continuation of the period matrix results in the right multiplication by the monodromy matrices M_γ,

$$\Delta_\gamma X(z) = X(z) \cdot M_\gamma. \tag{26.50}$$

Proposition 26.44. *If f is a polynomial transversal to infinity, then the determinant* $\det X(z)$ *of any period matrix, regardless of the choice of the forms $\boldsymbol{\omega}$, is a polynomial in z with zeros at the points of Σ.*

Proof. Together with Theorem 26.26, Proposition 26.43 implies that the determinant $\det X(z)$ is a single-valued function on $\mathbb{C} \smallsetminus \Sigma$. As z tends to infinity, the integrals occurring as the entries of $X(z)$ grow no faster than polynomially in $|z|$ in any sector. This implies that $\det X(z)$ is a polynomial. As z tends to a point $a \in \Sigma$, at least one cycle (a linear combination of the basis $\boldsymbol{\delta}(z)$) vanishes, hence the determinant tends to zero. □

From Proposition 26.44 it follows that (in the standing assumption that f is a Morse polynomial transversal to infinity) the determinant of *any* period matrix is divisible by the discriminant polynomial $D_f(z) = \prod_{z_j \in \Sigma}(z - z_j)$. The natural question is whether this description is precise, i.e., whether there are additional points at which the determinant of periods vanish. The answer is given by the following result.

Theorem 26.45. *Assume that the polynomial $f = f_{n+1} + \cdots$ of degree $n+1$ is transversal to infinity and the tuple of 1-forms $\boldsymbol{\omega} = (\omega_1, \dots, \omega_m)$ satisfies the assumptions of Theorem 26.21* (*their differentials generate the quotient* *algebra* $Q_{df_{n+1}} = \Lambda^2/df_{n+1}\wedge\Lambda^1$). *Then*

$$\det X(z) = c(\boldsymbol{\omega}) D_f(z), \qquad D_f(z) = \prod_{z_j\in\Sigma}(z-z_j), \quad c(\boldsymbol{\omega})\neq 0. \tag{26.51}$$

Proof. By the de Rham theorem [**War83**], for any basis of the homology of an arbitrary level curve L_a, $a\notin\Sigma$, there exist m forms on L_a such that the respective period matrix is nondegenerate. These m forms on L_a are restrictions of some tuple of polynomial 1-forms $\Omega_1,\dots,\Omega_m\in\Lambda^1[\mathbb{C}^2]$.

By Theorem 26.21, there exists a polynomial $m\times m$-matrix $P(z)$ of coefficients of expansion of Ω_j in the basis $\boldsymbol{\omega}$ of the module $\mathbf{P}_f$, such that

$$Y(z) = P(z)X(z), \qquad Y(z) = \begin{pmatrix} \oint\limits_{\delta_1(z)}\Omega_1 & \dots & \oint\limits_{\delta_m(z)}\Omega_1 \\ \vdots & \ddots & \vdots \\ \oint\limits_{\delta_1(z)}\Omega_m & \dots & \oint\limits_{\delta_m(z)}\Omega_m \end{pmatrix}. \tag{26.52}$$

The matrix $Y(a)$ is nondegenerate by construction, therefore $X(a)$ must also be nondegenerate. We conclude that the determinant $\det X(z)$ is a polynomial *without roots* outside Σ. Such a polynomial can be only of the form (26.51). □

Remark 26.46. In most expositions, Theorem 26.45 together with the Proposition 26.44, established by analytical and topological arguments, is the starting point of the construction of a basis for the module of Abelian integrals.

The usual strategy of proving the formula (26.51) is to compute the sectorial growth rate of all entries of the matrix $X(z)$ and show that $\det X(z) = O(|z|^n)$ as $|z|\to\infty$. This shows that $\det X(z)$ is a polynomial with roots at all n points of the critical locus Σ, which leaves the only possibility $\det X(z) = c\,D_f(z)$, where $c = c(\boldsymbol{\omega})\in\mathbb{C}$ is a constant. However, the accurate proof that $c\neq 0$ requires some effort; see [**Nov02**] where all details are explicitly supplied. The explicit value of $c(\boldsymbol{\omega})$ was recently obtained by A. Glutsyuk [**Glu06**]. For earlier results; see [**Var89**].

Staring from Theorem 26.45, one can derive (using the Cramer rule for finding the indeterminate coefficients), that the integral of any polynomial form Ω over any cycle $\delta(z)$ is a polynomial combination of integrals of the basic forms $\omega_1,\dots,\omega_m$ over this cycle, the coefficients being independent of the choice of the cycle,

$$\oint_{\delta(z)}\Omega = \sum_{j=1}^m p_j(z)\oint_{\delta(z)}\omega_j.$$

By Theorem 26.13, the 1-form $\Omega - \sum(p_j\circ f)\,\omega_j\in\Lambda^1$ is algebraically relative cohomologous to zero. This gives an alternative (analytic) proof of Theorem 26.21. For details see [**Gav98, Nov02**].

We choose an alternative strategy based on Theorem 26.21, since it is algorithmic and provides explicit bounds for the operator of decomposition of any polynomial 1-form in the elements of the basis for $\mathbf{P}_f$.

26K. Monodromy of Abelian integrals. Monodromy of the Picard–Fuchs system (26.41) is a linear representation of the fundamental group $\pi_1(\mathbb{C} \smallsetminus \Sigma, a)$ by automorphisms of solutions of the system. It turns out that the easiest way to study this representation is via topology of the map $f\colon \mathbb{C}^2 \to \mathbb{C}$.

26$\mathbf{K}_1$. *Completeness of the system of vanishing cycles.* A generic (Morse) polynomial of degree $n+1$ transversal to infinity, has n^2 critical points and hence every nonsingular level curve carries exactly n^2 (topological continuations) of vanishing cycles. On the other hand, the homology group of a generic leaf L_z also is of rank n^2 over $\mathbb{Z}$, as shown in Theorem 26.31. The two numbers are equal, and this equality is not accidental.

Theorem 26.47. *Vanishing cycles generate the first homology group of any fiber $L_z = \{f = z\} \subset \mathbb{C}^2$ of a Morse polynomial transversal to infinity.*

Clarification and references. The precise meaning of this theorem is as follows. For a fixed regular value $a \in \mathbb{C} \smallsetminus \Sigma$ of f consider simple paths α_j, $j = 1, \dots, m$, connecting a with each of the critical values $a_1, \dots, a_m$ of f, $m = n^2$. Each vanishing cycle $\delta_j \in H_1(L_z, \mathbb{Z})$ is well defined for all $z \in (\mathbb{C}, a_j)$ and can be uniquely continued along α_j^{-1} to a cycle $\delta_j \in H_1(L_a, \mathbb{Z})$. Besides, each path α_j defines a loop $\gamma_j \in \pi_1(\mathbb{C} \smallsetminus \Sigma, a)$ which corresponds to going along α_j, encircling a_j by a sufficiently small positive circular loop and returning back along the same path α_j^{-1} (i.e., inverting the direction). The collection of paths $\{\alpha_1, \dots, \alpha_m\}$ is called *proper*, if the simple loops $\gamma_1, \dots, \gamma_m$ generate the fundamental group $\pi_1(\mathbb{C} \smallsetminus \Sigma, a)$.

Theorem 26.47 asserts that for any polynomial f and any regular value a one can always construct a proper system of paths $\alpha_1, \dots, \alpha_m$ such that the corresponding continuations of vanishing cycles $\delta_1(a), \dots, \delta_m(a)$ generate the entire homology $H_1(L_a, \mathbb{Z})$.

This assertion can be derived from the corresponding local result, Theorem 1 from [**AGV88**, Chapter I, §2]. Alternatively, one can use the results by A. B. Zhizhchenko [**Žiž61**]. A very good exposition of these things is given in the recent paper [**MMJR97**, Sect. 4]: Theorem 26.47 is an immediate corollary to the Theorem 4.4 of the latter paper. □

26$\mathbf{K}_2$. *Transitivity of the monodromy on vanishing cycles.* The global structure of the topological monodromy group is characterized by the following property.

Theorem 26.48. *The monodromy group acts transitively on the collection of all vanishing cycles: for any two such cycles $\delta_1, \delta_2 \in L_a$, $a \notin \Sigma$, there exists a loop $\gamma \in \pi_1(\mathbb{C} \smallsetminus \Sigma, a)$ such that $\Delta_\gamma \delta_1 = \pm\delta_2$.*

Proof. Assertion of this theorem follows from the fact that the discriminant variety $\boldsymbol{\Sigma}$, introduced in (26.35), is irreducible and hence its smooth part, parameterizing in a certain sense all vanishing cycles of all polynomials with a fixed principal homogeneous part, is connected. We follow the exposition in [**Pus97**]; see also [**AGV88**, Theorem 4, Chapter I, §3].

Let $f \in \mathbb{C}[x, y]$ be a Morse polynomial transversal to infinity, and $a_1, a_2 \in \Sigma$ the critical values, $a_i = f(x_i, y_i)$, corresponding to the two cycles δ_i vanishing at the two critical points $C_i = (x_i, y_i) \in \mathbb{C}^2$, $i = 1, 2$. Theorem 26.48 will be proved, if we find a path γ_{12} connecting a_1 with a_2 and avoiding Σ everywhere else, such that the parallel transport (continuation) of $\delta_1(z)$, $z \in (\mathbb{C}, a_1)\smallsetminus\Sigma$, along this path coincides with $\delta_2(z)$, $z \in (\mathbb{C}, a_2)\smallsetminus\Sigma$, modulo orientation of the latter.

We will show first that there exists a continuous deformation of the (singular) zero level curve $L_{a_1} = \{f - a_1 = 0\}$ onto the other singular level curve $L_{a_2} = \{f - a_2 = 0\}$, which sends the respective (uniquely defined) critical points into each other, if one is allowed to change continuously *all nonprincipal coefficients* of the polynomial f rather than only its free term.

To that end, consider the universal deformation $\Phi(\lambda; x, y) = f_{n+1} + \sum_{0\leqslant i+j\leqslant n} \lambda_{ij} x^i y^j$ as in (26.34) and the discriminant variety $\boldsymbol{\Sigma}$ introduced in (26.35). Let $\boldsymbol{\Sigma}^\circ \subset \boldsymbol{\Sigma}$ be the set of parameters $\lambda \in P \cong \mathbb{C}^r$ such that the zero level curve of the polynomial $f_\lambda = \Phi(\lambda; \cdot, \cdot)$ carries only one nondegenerate critical point. This is the principal stratum of the algebraic variety $\boldsymbol{\Sigma}$, a relatively open subset with a complement which is an algebraic variety of lower dimension.

Lemma 26.49.

1. *All points of $\boldsymbol{\Sigma}^\circ$ are smooth on $\boldsymbol{\Sigma}$.*
2. *The set $\boldsymbol{\Sigma}^\circ$ is connected.*

Proof of the lemma. To prove the first assertion, consider the polynomial $\Phi' = \Phi - \lambda_{00}$ which in fact depends only on the parameters λ_{ij} with $i+j > 0$. By the implicit function theorem, Morse critical points are *stable*: if $\lambda \in \boldsymbol{\Sigma}^\circ$ and $C_* \in \mathbb{C}^2$ is the corresponding Morse critical point of $\Phi'|_\lambda$, then for any sufficiently close combination of the parameters $\lambda' = \{\lambda_{ij},\ i + j > 0\}$, the polynomial $f'_{\lambda'} = \Phi'(\lambda', \cdot, \cdot)$ has a nearby Morse critical point $C(\lambda')$ analytically depending on λ'. The critical value $s(\lambda')$ of $f'_{\lambda'}$ at this point will also depend analytically on λ', which means that $\boldsymbol{\Sigma}^\circ$ is the graph of an analytic function, $(\lambda', -s(\lambda')) \in \boldsymbol{\Sigma}^\circ$.

To prove connectedness of $\boldsymbol{\Sigma}^\circ$, note that $\boldsymbol{\Sigma}$ is the image of the surface

$$S = \{(\lambda, x, y) \in \mathbb{C}^r \times \mathbb{C}^2 \colon \partial\Phi/\partial x = \partial\Phi/\partial y = \Phi = 0\} \tag{26.53}$$

by the projection $(\lambda, x, y) \mapsto \lambda$. The part S' of S given by the inequality $\{\det\big(\partial^2\Phi/\partial(x,y)\big) \neq 0\}$, by the first assertion of the lemma, parameterizes smooth points of $\boldsymbol{\Sigma}$, including self-intersections of several smooth components.

The projection $\pi\colon (\lambda, x, y) \mapsto (x, y)$ restricted on S is a holomorphic *affine* subbundle of the trivial bundle $\pi\colon (\lambda, x, y) \mapsto (x, y)$. Indeed, the equations (26.53) for any fixed (x, y) are affine with respect to λ and define an affine subspace in P. The local triviality follows from the fact that any translation in the (x, y) plane corresponds to an affine transformation of the parameters λ: the nonprincipal coefficients of the polynomial $\Phi(\lambda; x{+}a, y{+}b)$ for any $(a, b) \in \mathbb{C}^2$ are affine functions of λ.

The degeneracy condition $\det\big(\partial^2 f/\partial(x,y)^2\big) = 0$ as well as the occurrence of another critical point determine a *proper* complex affine subbundle in S. This properness guarantees that the complementary set

$$S^\circ = \big\{(\lambda, x, y) \in S\colon \det\big(\partial^2 f/\partial(x,y)^2\big) \neq 0,\ \forall (x', y') \in \mathbb{C}^2,\ (\lambda, x', y') \notin S\big\}$$

which parameterizes $\boldsymbol{\Sigma}^\circ$, is connected. The lemma is proved. □

We return to the proof of Theorem 26.48. Let $\mathbf{a}_1, \mathbf{a}_2 \in \boldsymbol{\Sigma} \subset \mathbb{C}^r$ be the points corresponding to the nonprincipal coefficients of the polynomials $f - a_1$ and $f - a_2$ respectively, where $f = f_{n+1} + \cdots$ is the initial polynomial. By Lemma 26.49, these points appear at the intersection of the smooth part $\boldsymbol{\Sigma}^\circ$ with the complex line $\ell = \{\lambda' = \text{const}\}$.

Since $\boldsymbol{\Sigma}^\circ$ is connected, the points $\mathbf{a}_1$ and $\mathbf{a}_2$ can be connected by a path $\boldsymbol{\gamma}_{12}$ parameterized by $t \in [1, 2]$. Since $\boldsymbol{\Sigma}^\circ$ is a smooth hypersurface, this path can be deformed to a path that avoids $\boldsymbol{\Sigma}$ everywhere except for the endpoints, but remains sufficiently close to $\boldsymbol{\Sigma}$ so that the vanishing cycle on all the level curves $L_t = L_{\boldsymbol{\gamma}_{12}(t)} \subseteq \mathbb{C}^2$ is uniquely determined. The corresponding deformation of zero level curves of the polynomial $\Phi(\boldsymbol{\gamma}_{12}(t), \cdot, \cdot)$ carries the unique vanishing cycle δ_1 on $\{\Phi(\mathbf{a}_1, \cdot, \cdot) = 0\}$ onto the unique vanishing cycle δ_2 on $\{\Phi(\mathbf{a}_1, \cdot, \cdot) = 0\}$.

During this deformation all nonprincipal coefficients of the polynomial are changed, not just the free term. However, by the global Zariski theorem, the path $\boldsymbol{\gamma}_{12} \subset \mathbb{C}^r \smallsetminus \boldsymbol{\Sigma}$ with the endpoints on the line ℓ can be deformed (by a homotopy with the fixed endpoints) to a path γ_{12} entirely belonging to $\ell \smallsetminus \boldsymbol{\Sigma} = \ell \smallsetminus \Sigma$. In this deformation only the free term of f is changed, hence the corresponding deformation coincides with the monodromy action described in Corollary 26.28. The proof of Theorem 26.48 is complete. □

26$\mathbf{K}_3$. *Almost irreducibility of the monodromy.* The transitivity established in Theorem 26.48, allows us to prove that the topological monodromy group is almost irreducible in the following precise sense. Consider the first homology $G = H_1(L_a, \mathbb{Z})$ of a generic level curve L_a, $a \notin \Sigma$. This is a $\mathbb{Z}$-module equipped with an antisymmetric intersection form $G^2 \ni (\delta, \delta') \mapsto \delta \cdot \delta' \in \mathbb{Z}$. Since L_a is $n+1$ times punctured Riemann surface of some genus g, the module G is generated by $2g$ canonical cycles ℓ_j, ℓ'_j, $j = 1, \dots, g$ and any n "small loops" $s_1, \dots, s_n$ around $n+1$ punctures at infinity. The intersection form in this basis looks very simple:

$$\ell_i \cdot \ell'_j = \begin{cases} 1, & i = j, \\ 0, & i \neq j, \end{cases} \qquad s_k \cdot G = 0. \tag{26.54}$$

Any loop $\gamma \in \pi_1(\mathbb{C} \smallsetminus \Sigma, a)$ in the base defines a topological monodromy operator $\Delta_\gamma \colon G \to G$ which is $\mathbb{Z}$-linear and preserves the intersection form. Together the operators determine $\mathbb{Z}$-linear representation $\gamma \mapsto \Delta_\gamma$ of the fundamental group by automorphisms of the module G. If $S \subseteq G$ is a submodule *invariant by all monodromy operators*, the quotient representation by automorphisms of the quotient module G/S is well defined by the action

$$\widetilde{\Delta}_\gamma(\delta \bmod S) = (\Delta_\gamma \delta) \bmod S, \qquad \forall (\delta \bmod S) \in G/S.$$

Theorem 26.50. *For a polynomial f of degree $n+1$ transversal to infinity, the submodule $S \subset G = H_1(L_a, \mathbb{Z})$ generated by the "small loops" $s_1, \dots, s_n$ is invariant and each monodromy operator Δ_γ acts identically on it.*

The action of the topological monodromy $\gamma \mapsto \widetilde{\Delta}_\gamma \in \operatorname{Aut}(G/S)$ on the quotient $\mathbb{Z}$-module G/S is irreducible, i.e., has no nontrivial invariant submodules.

Proof. The first assertion follows from the Picard–Lefschetz formulas (26.47) and the structure of the intersection form (26.54): the "small loops" s_k have zero intersection index with any vanishing cycle.

To prove the second assertion, consider an arbitrary invariant submodule $G' \neq 0 \bmod S$ in G/S. By factoring out all "small loops" the quotient G/S inherits the intersection form which is *nondegenerate*: if a cycle δ is orthogonal to all G (i.e., has zero intersection index with all loops ℓ_j, ℓ'_j for all $j = 1, \dots, g$), then δ itself is zero modulo S.

We first claim that G' contains one of the vanishing cycles $\bmod S$. Indeed, an element $\delta \notin S$ cannot be orthogonal to all vanishing cycles: since the latter generate the entire G, this would contradict to the nondegeneracy of the intersection form on G/S. If the intersection index $\delta \cdot \delta_i \neq 0$, then together with δ the submodule G' also contains the element $\delta + c\delta_i$ with $c \neq 0$ because of the invariance and the Picard–Lefschetz formula (26.47). Then $\delta_i \in G' \bmod S$. But because of the transitivity of the action of the topological monodromy (Theorem 26.48), G' contains all other vanishing cycles.

This means that $G' = G \bmod S$, that is, a nonzero invariant submodule necessarily coincides with the entire module G. This proves the irreducibility of the quotient action. □

Since the topological monodromy acts on the period matrix $X(z)$ by right multiplications as in (26.50), it coincides with the monodromy of the corresponding Picard–Fuchs system (26.41), and we obtain a rather important property of the latter system.

Corollary 26.51. *The monodromy group of the Picard–Fuchs system* (26.41) *is almost irreducible*: *it has an n-dimensional subspace on which all monodromy operators are identical, while the quotient representation $\gamma \mapsto \widetilde{\Delta}_\gamma$ is irreducible.* □

26**K**$_4$. *Gauss–Manin connexion.* The first homology $H_1(L_a, \mathbb{Z})$ and cohomology $H^1(L_a, \mathbb{C})$ of a generic fiber $L_a = f^{-1}(a) \in \mathcal{F}$ are a lattice ($\mathbb{Z}$-module) and a complex space of the rank (resp., complex dimension) equal to n^2. To achieve uniformity, we consider the linear spaces $H_1(L_a, \mathbb{C}) = H_1(L_a, \mathbb{Z}) \otimes \mathbb{C}$ of formal combinations of cycles with complex coefficients. Then we obtain two families of complex spaces of the same dimension, indexed by nonsingular values $a \notin \Sigma$.

Each of these families is a holomorphic vector bundle over the set of regular values $X = \mathbb{C} \smallsetminus \Sigma$, equipped with meromorphic connexions. To see this, consider first the *homology bundle*: because of the local topological triviality (Theorem 26.26), we can choose any basis in the homology of a nonsingular fiber and then carry it continuously to all nearby regular fibers. This gives local trivialization of the homology bundle $H_1(\cdot, \mathbb{C}) \to X$ and a locally flat connexion $\nabla_\circ$ on it: sections horizontal in the sense of this connexion are continuous sections of the projection $H_1(\cdot, \mathbb{Z}) \to X$.

To define local trivializations of the *cohomology bundle* $H^1(\cdot, \mathbb{C}) \to X$, note that any polynomial 1-form $\omega \in \Lambda^1[\mathbb{C}^2]$ defines a section $[\omega]\colon X \to H^1(\cdot, \mathbb{C})$: for any point $z \in X$, the value $[\omega](z)$ is the cohomology class of the form ω restricted on the leaf $L_z \subset \mathbb{C}^2$. For any $n^2 = m$ 1-forms $\omega_1, \dots, \omega_m$, whose restrictions on a given nonsingular fiber L_a are cohomologically independent, the independence also persists on all nearby fibers L_z for all $z \in (X, a)$. The corresponding sections $[\omega_1], \dots, [\omega_m]$, clearly locally holomorphic, furnish a local trivialization of the cohomology bundle $H^1(\cdot, \mathbb{C}) \to X$.

Integration $(\alpha, \delta) \mapsto \oint_\delta \alpha$ is a natural pairing (duality) between the two line bundles over $X = \mathbb{C} \smallsetminus \Sigma$. This duality allows us to carry different structures from one bundle to the other. In particular, there is a natural connexion ∇° on the cohomology bundle, dual to the $\nabla_\circ$ on the homology bundle. This connexion, called the *Gauss–Manin connexion*, is uniquely determined the identity

$$d(\alpha, \delta) = (\alpha, \nabla_\circ \delta) + (\nabla^\circ \alpha, \delta) \qquad \forall \alpha\colon X \to H^1(\cdot, \mathbb{C}),\ \delta\colon X \to H_1(\cdot, \mathbb{C})$$

valid for any two sections α and δ of the homology and cohomology bundles respectively. Choosing a basis of *horizontal* sections $\delta_1(z), \dots, \delta_m(z)$ with $\nabla_\circ \delta_j = 0$, we see that for the sections $[\omega_1], \cdots, [\omega_m]$ their covariant derivatives are expansions of the *differential* of the period matrix $(\omega_i, \delta_j(z))$ in the periods of the forms ω_j themselves. In other words, the Picard–Fuchs system of linear equations (26.41) in this geometric language is nothing more than the matrix form of the Gauss–Manin connexion with respect to the chosen trivialization of the cohomology bundle.

Note that though the form ω is “constant” (does not explicitly depend on z), its restriction on L_z is not constant in the sense of the Gauss–Manin connexion, i.e., the section $[\omega]$ is not horizontal.

26L. Real branches of Abelian integrals and lower bounds for the number of limit cycles. Now we can return to the Example 26.11 and show that this type of behavior of Abelian integrals is impossible if f is a Morse polynomial transversal to infinity.

Theorem 26.52 (Yu. Ilyashenko [**Ily69**], I. Khovanskaya (Pushkar′) [**Pus97**]). *Let $f \in \mathbb{R}[x,y]$ be a real polynomial transversal to infinity whose complexification is a Morse function, and $\gamma(t)$ a continuous family of real ovals on the level curves of f.*

If ω is an arbitrary polynomial 1-form with identically zero integral over $\gamma(t)$, then ω is relatively exact (can be represented as in (26.18)). If $\deg\omega < \deg df$, then ω is exact, $\omega = dg$, $g \in \mathbb{R}[x,y]$.

To prove this result, we need the following topological lemma.

Lemma 26.53. *Any nonsingular real oval on the level curve of a real Morse polynomial, either is itself the continuation of a vanishing cycle, or has a nonzero intersection index with at least one vanishing cycle.*

Proof. Any real oval $\gamma \subset \mathbb{R}^2$ belongs to a topological annulus on the plane filled by real ovals of level curves. The inner boundary of this annulus cannot be empty. If the inner boundary is a critical point of f, then γ itself is a vanishing cycle. Otherwise the inner boundary is a singular oval carrying a (Morse) critical point of f which is a saddle. The cycle vanishing at this saddle is purely imaginary and intersects γ with the index ±1. □

Proof of the Theorem 26.52. Suppose that $I(t) = \oint_{\gamma(t)} \omega \equiv 0$. By the Picard–Lefschetz formula (26.47) and Lemma 26.53, for at least one vanishing cycle $\delta_1(z)$ the integral $I_1(z) = \oint_{\delta_1(z)} \omega$ is also vanishing identically. But then by Theorem 26.48, the integral of ω over *any* vanishing cycle of f is identically zero. Since vanishing cycles generate the homology group of any fiber (Theorem 26.47), this means that the 1-form ω is *relatively closed*.

Application of Theorem 26.13, whose assumptions are automatically satisfied if f is a Morse polynomial transversal to infinity, shows then that ω is (algebraically) relatively exact: $\omega = h\,df + dg$ with *some* polynomials h, g. Symmetrizing this identity (adding it with its complex conjugate), we can assume without loss of generality that both h, g are real.

To prove the second assertion of the theorem, note that the polynomial "primitive" g obtained by the integral (26.19) of a form of degree $\deg\omega < \deg df$ grows no faster than $o(|x|+|y|)^{n+1}$ (as follows from direct estimates) and hence is a polynomial form of degree not exceeding n. The difference $\omega - dg$ is a polynomial 1-form divisible by the form df of a *higher* degree. This means that $h = 0$ and ω is exact, $\omega = dg$. □

Remark 26.54. The second assertion of Theorem 26.52 can be proved using a more direct argument as in [**Pus97**]. Consider the Gelfand–Leray derivative $\frac{d\omega}{df}$ of the form ω. By Remark 26.34, this derivative has poles of order at most 1 at infinity. If the integral of ω along any cycle on L_z is zero, then the residues at these points are all zeros. Since the poles are simple, absence of the residues means that the derivative $\frac{d\omega}{df}$ is in fact holomorphic at these points and its primitive on each compactified fiber $\overline{L_z}$, is a holomorphic function, which is necessarily a constant. Hence the derivative $\frac{d\omega}{df}$ is itself zero restricted on each fiber, and thus $d\omega = 0$.

As an application of Theorem 26.52, we construct a polynomial foliation of the plane from the class $\mathcal{A}_n$ having $\frac{1}{2}(n+1)(n-2)$ limit cycles.

Theorem 26.55 (Yu. Ilyashenko, [**Ily69**], I. Khovanskaya (Pushkar′), [**Pus97**])**.** *If $f \in \mathbb{R}[x,y]$ is a Morse polynomial of degree $n+1$ transversal to infinity, then for any $N = \frac{1}{2}(n+1)(n-2)$ real ovals of the integrable foliation $\{df = 0\}$ on $\mathbb{R}^2$ one can construct a form*

$$\omega = P(x,y)\,dx + Q(x,y)\,dy, \qquad P,Q \in \mathbb{R}[x,y], \quad \deg P,Q \leqslant n, \quad (26.55)$$

such that the perturbation $\{df + \varepsilon\omega = 0\}$ (cf. with (26.1)) produces at least N limit cycles which converge to the specified ovals as $\varepsilon \to 0$.

Proof. Consider the linear space of all polynomial 1-forms ω of the specified degree: the dimension of this space is $n(n+1)$. The exact forms constitute a subspace of dimension $\frac{1}{2}(n+2)(n+1) - 1$ in it. The quotient space has dimension $n(n+1) - \frac{1}{2}(n+2)(n+1) + 1$ which is exactly equal to $N+1$.

For any choice of N real ovals $\delta_i \subseteq \{f = c_i\}$, $i = 1,\dots,N$, the condition $\oint_{\delta_i} \omega = 0$ constitutes a linear restriction on the form ω. As soon as the number of restrictions is less than the dimension of the (quotient) space, there exists at least one form, by construction *not exact*, whose integral is zero along all the specified ovals.

By Theorem 26.52, *all these zeros are isolated*. Indeed, otherwise the integral must have a real branch which is identically zero, which is possible only if the form is exact.

If all these zeros are simple, then by Remark 26.2, the corresponding perturbation will produce at least N limit cycles.

If some of the zeros are of even orders, then the corresponding limit cycles can "escape" into the nonreal domain. In this case the perturbation form should be produced in the following way. Assume without loss of generality that all ovals $\delta_1,\dots,\delta_N$ are oriented positively so that the form $\omega_0 = y\,dx$ has *negative* integral over each such oval. Let k_+ denote the number of ovals such that the corresponding real branch of the integral $\oint \omega$ has a local

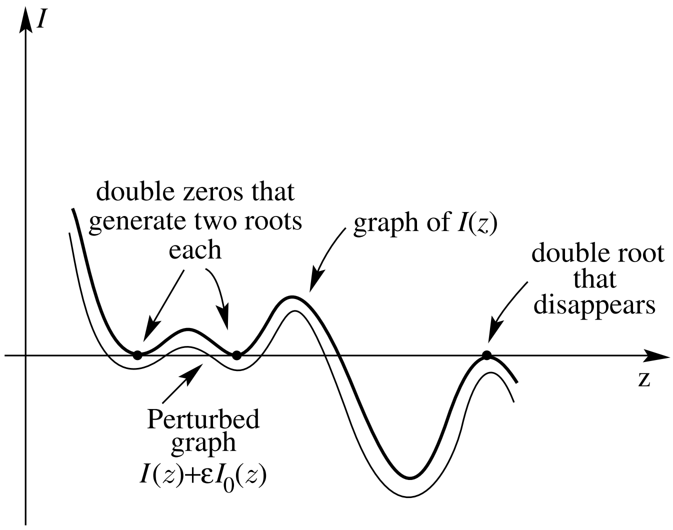

Figure V.5. Construction of the perturbation with the specified number of simple roots

minimum there, and by k_- the number of ovals yielding the local maximum (recall that in any case the roots are isolated). The total number $k_+ + k_-$ is equal to the number of different ovals where the integral $I(z) = \oint_{\delta(z)} \omega$ has a local extremum. The remaining $N - (k_+ + k_-)$ ovals correspond to the roots where I changes its sign. If $k_+ \geqslant k_-$, consider the form $\omega + \epsilon\omega_0$, where $1 \gg \epsilon > 0$ is an auxiliary parameter: the corresponding integral is obtained by subtracting a small everywhere positive quantity $I_0(z) = \oint_{\delta(z)} \omega_0$ from the function $I(z)$.

In particular, each of the k_+ local minima of I will produce at least two odd order roots, roots at the local maxima will disappear, and every odd order root from the remaining $N - (k_+ + k_-)$ will produce at least one odd order root again if ϵ is sufficiently small; see Fig. V.5. Since $k_+ \geqslant k_-$, the total order of odd order roots that appear after this small variation *of the form* $\omega_\epsilon = \omega + \epsilon y\,dx$ of the corresponding integral $I(z) + \epsilon I_0(z)$ will have at least $N - (k_+ + k_-) + 2k_+ \geqslant N$ odd order roots. The same Remark 26.2 shows now that the number of limit cycles in this degenerate case will be again no less than N. □

Remark 26.56. The lower bound for the number of zeros of Abelian integrals (and the respective limit cycles) is not sharp. N. F. Otrokov constructed in [**Otr54**] examples with a larger number of limit cycles. However all of them encircle a unique singular point of the foliation, whereas Theorem 26.55 allows to place them much more freely.

The principal term of both the Otrokov's lower bound and the bound achieved in Theorem 26.55 is of the same form $\frac{1}{2}n^2 + O(n)$. However, for special *rather symmetric* polynomials f of degree $n+1$ one can construct Abelian integrals of forms of degree n having more zeros. For instance, there are known examples with as many as $n^2 - 1$ isolated real zeros for $n = 2, \dots, 10$.

Exercises and Problems for §26.

Problem 26.1. Prove that a real foliation is really analytically integrable near an identical cycle.

Exercise 26.2. Why Proposition 26.1 does not apply to a neighborhood of a *critical* level curve $\{f = 0\}$, carrying a critical point?

Problem 26.3. Prove that a continuous branch of any real Abelian integral (over a continuous family of compact ovals of f) is real analytic on any interval free from real critical values of the polynomial f.

Problem 26.4. Let γ be a real oval of a cubic ultra-Morse polynomial. Prove that for any ε there exists a quadratic vector field with a limit cycle of multiplicity 2 whose Hausdorff distance from γ is smaller than ε.

Problem 26.5. Prove that for any r there exists a real polynomial vector field from the class $\mathcal{A}_r$ with a limit cycle of multiplicity $\frac{1}{2}(r+1)(r-2)$.

Problem 26.6. Prove that the Bonnet set of any polynomial is an algebraic subset in $\mathbb{C}$. Give an example of a polynomial $f \in \mathbb{C}[x, y]$ with an infinite Bonnet set $\operatorname{Bs}(f) = \mathbb{C}$.

Problem 26.7. Prove Proposition 26.17.

Exercise 26.8. Find atypical values of the polynomial $f(x, y) = xy(xy - 1)$.

Problem 26.9. Prove that for polynomials not transversal to infinity, the period matrix remains single-valued and has at worst poles at the atypical values.

Problem 26.10. Prove that for a polynomial $f = f_{n+1} + \cdots$ transversal to infinity, the level curves $f = z$ and $f_{n+1} = z$ are homeomorphic for all sufficiently large $|z|$ (cf. with the end of the proof of Theorem 26.31).

Problem 26.11. Consider a compact Riemann surface C and a noncontractible simple loop γ on it. Because of the noncontractibility, the difference $C \smallsetminus \gamma$ is connected and has a boundary which is homeomorphic to two disjoint circles. Sealing the two holes by topological disks results in a new surface $\widetilde{C} = C/\gamma$ called *pinching* of C along γ.

Compare the Euler characteristic of C and C/γ.

Problem 26.12. If f is a complex polynomial transversal to infinity with only nondegenerate critical points (some of the critical values can coincide), then any critical level curve can be obtained from a nearby nonsingular level curve by pinching along the corresponding vanishing cycles. Prove this statement.

Problem 26.13. Prove the Plücker formula (25.27), using Problems 26.11 and 26.12.

Problem 26.14. For any collection of cycles $c_1, \ldots, c_m$ on a Riemann surface their *intersection graph* is a graph with m vertices which are connected by an edge if and only if the corresponding intersection index $c_i \cdot c_j$ is nonzero.

Prove that for any polynomial transversal to infinity, one can construct a basis of vanishing cycles (as in the Clarification to Theorem 26.47) such that the corresponding intersection graph is connected.

Hint. Consider a small perturbation of the degenerate polynomial $f = \prod_{j=1}^{n+1} l_j$ with generic affine polynomials of degree 1.

Problem 26.15. Prove that for *any* polynomial and *any* proper system of paths the intersection graph constructed in Problem 26.14 is always connected.

Problem 26.16. Derive from the Problems 26.14 and 26.15 the assertion on transitivity of action (Theorem 26.48).

The Problems 26.17–26.22 together give an upper bound for the *multiplicity* of an isolated zero of an Abelian integral. This result was obtained by P. Mardešić [**Mar91**].

Problem 26.17. Let f be a Morse polynomial of degree $n+1$ transversal to infinity, and $\omega \in \Lambda^1[\mathbb{C}^2]$ a polynomial form of degree n. Consider the integrals $J_k(z) = \oint_{\delta_k(z)} \omega$, where $\delta_1(z), \ldots, \delta_m(z)$ are vanishing cycles constructed as in Theorem 26.47. Let $W(z)$ be a Wronski determinant of the functions $J_1, \ldots, J_m$.

Prove that $W(z)$ is a rational function of z and describe its polar locus.

Problem 26.18. Prove that $W \equiv 0$ if and only if ω is closed.

Hint.: Use the Problem 26.15 and exactness theorem.

Problem 26.19. Estimate from above the order of a pole of W at any critical point of the ultra-Morse polynomial.

Problem 26.20. Prove that the integral J_k in Problem 26.17 has an algebraic singular point at infinity: a branch point of order equal either to $r+1$, or to a divisor of $r+1$.

Problem 26.21. Give an upper bound for the order of the pole of W at infinity.

Problem 26.22. Give an upper bound for the *multiplicity* of an isolated zero of any of the integrals $J_k(z)$.

Problem 26.23. Consider a real ultra-Morse polynomial H with a compact component Γ of a critical level that contains a critical point A and is not a singleton. Prove that this component is an eight shaped figure. Let the corresponding critical value be zero, and a level curve $\{H = \varepsilon\}$ has a smooth component Γ_ε such that Γ lies inside Γ_ε for any small positive ε. Then the level curve $\{H = -\varepsilon\}$ has two components for the same ε, denoted by $\Gamma^1_{-\varepsilon}$ and $\Gamma^2_{-\varepsilon}$, one contained in one loop of Γ, another in another one. Let $\delta_\varepsilon \subset \{H = \varepsilon\}$ be a vanishing cycle close to A. Consider a loop $\Gamma_{-\varepsilon} \subseteq L_{-\varepsilon} = \{H = -\varepsilon\}$ obtained from Γ_ε by continuation over a half-circle of radius ε centered at zero in the set of noncritical values of

H. Find the expression of the corresponding element $[\Gamma_{-\varepsilon}] \in H_1(L_{-\varepsilon}, \mathbb{Z})$ through $[\Gamma^1_{-\varepsilon}], [\Gamma^2_{-\varepsilon}], [\delta_{-\varepsilon}]$.

Problem 26.24. Prove that for any r there exists a real polynomial vector field from the class $\mathcal{A}_r$ with a limit cycle of multiplicity $\frac{1}{2}(r+1)(r-2)$.

Problem 26.25. Modifying the proof of Theorem 26.35, write explicitly the Picard–Fuchs system for the cubic polynomial $f(x,y) = \frac{1}{2}(x^2+y^2) - \frac{x^3}{3}$ and the two forms $\omega_1 = y\,dx$ and $\omega_2 = xy\,dx$.

Problem 26.26. Let $f(x,y) = y^2 + x^{n+1} + p_n(x)$ be a *hyperelliptic* polynomial. Prove that the forms $y\,dx, xy\,dx, \dots, x^{n-1}y\,dx$ form a basis of the corresponding Petrov module $\mathbf{P}_f$. Modifying the proof of Theorem 26.35, write explicitly the Picard–Fuchs system for the corresponding *hyperelliptic integrals*.

27. Topological classification of complex linear foliations

The famous Grobman–Hartman theorem [**Gro62, Har82**] asserts that any real vector field whose linearization matrix is hyperbolic (i.e., has no eigenvalues with zero real part), is topologically orbitally equivalent to its linearization. An elementary analysis shows that two hyperbolic linear real vector fields are orbitally topologically conjugated if and only if they have the same number of eigenvalues to both sides of the imaginary axis.

This section describes the complex counterparts of these results. From the real point of view a holomorphic 1-dimensional singular foliation on $(\mathbb{C}^n, 0)$ by phase curves of a holomorphic vector field is a 2-dimensional real analytic foliation on $(\mathbb{R}^{2n}, 0)$. If the singularity at the origin is in the Poincaré domain, this foliation induces a *nonsingular real* 1*-dimensional* foliation (*trace*) on all small $(2n-1)$-dimensional spheres $\mathbb{S}^{2n-1}_\varepsilon = \{|x_1|^2 + \cdots + |x_n|^2 = \varepsilon > 0\}$. Under the *complex* hyperbolicity-type conditions excluding resonances, the trace is generically structurally stable. Poincaré resonances manifest themselves via *bifurcations* of this trace foliation.

On the contrary, if the singularity is in the Siegel domain, the corresponding foliations exhibit *rigidity*: two foliations are topologically equivalent if and only if there is a rather special conjugacy between them which is completely determined by n complex numbers. This rigidity implies that there are *continuous invariants* (moduli) of topological classification.

27A. Trace of the foliation on the small sphere. Consider the real sphere of radius $\varepsilon > 0$,

$$\mathbb{S}_r = \{r^2(x) = \varepsilon\} \subseteq \mathbb{C}^n, \qquad r^2(x) = |x|^2 = \sum_1^n x_i \bar{x}_i. \tag{27.1}$$