Problem 6.3. Compute all holonomy maps of an integrable foliation $\{du = 0\}$, $u \in \mathcal{O}(\mathbb{C}^2, 0)$, if $u = \prod u_j^{p_j}$ is the primary decomposition of the holomorphic germ u with irreducible factors u_j and natural exponents $p_j \in \mathbb{N}$.

Problem 6.4. Prove that a formally integrable holomorphic self-map (or a finitely generated group G of holomorphic germs of self-maps from $\operatorname{Diff}(\mathbb{C}, 0)$) is also analytically integrable; cf. with Theorem 6.8.

Suggestion. Use the formal chart in which $\widehat{u}(z) = z^m$.

Problem 6.5. Prove that an (orbital) symmetry of a holomorphic vector field on $(\mathbb{C}, 0)$ is necessary holomorphic itself.

Problem 6.6. Construct a finitely generated subgroup $G \subset \operatorname{Diff}(\mathbb{C}, 0)$, whose orbits are dense in each of the two half-planes $\{\pm \operatorname{Im} z > 0\}$ separately, yet both half-planes are invariant by G.

Generalize this example and find a group whose orbits are dense in each of $2p$ invariant sectors in $(\mathbb{C}, 0)$ for any $p > 1$ (cf. with Theorem 6.53).

Problem 6.7 (formal rigidity of generic groups)**.** Assume that two finitely generated subgroups $G, G' \subseteq \operatorname{Diff}(\mathbb{C}, 0)$ are *formally* equivalent and one of these groups contains a hyperbolic germ. Prove that in such case G and G' are holomorphically equivalent, moreover, any formal conjugacy between them is necessarily holomorphic (convergent).

7. Holomorphic invariant manifolds

In this short section we show that under rather weak conditions one can eliminate enough nonresonant terms to ensure existence of *holomorphic invariant (sub)manifolds*. Recall that a holomorphic submanifold $W \subset (\mathbb{C}^n, 0)$ is invariant for a holomorphic vector field F, if the vector $F(x)$ is tangent to W at any point $x \in W$. Traditionally the prefix 'sub' is omitted, though it plays an important role: in §14 we will discuss invariant analytic subvarieties that are *not* submanifolds because of their singularity.

7A. Invariant manifolds of hyperbolic singularities. Suppose that the spectrum $S \subset \mathbb{C}$ of linearization matrix A of a holomorphic vector field consists of two parts $S^\pm \subset \mathbb{C}$ separated by a real line (i.e., each part belongs to an open half-plane bounded by the line). In this case no eigenvalue from one part can be equal to a linear combination of eigenvalues from the other part with nonnegative coefficients,

$$\begin{gathered} \lambda_j^- - \sum \alpha_i \lambda_i^+ \neq 0, \qquad \lambda_i^+ - \sum \alpha_j \lambda_j^- \neq 0, \\ \lambda_i^+ \in S^+, \quad \lambda_j^- \in S^-, \quad \alpha_i, \alpha_j \in \mathbb{Z}_+, \end{gathered} \tag{7.1}$$

(we say that there are no *cross-resonances* between the two parts). Without loss of generality A can be assumed to be in the block diagonal form. By the Poincaré–Dulac theorem, there exists a formal transformation eliminating all nonresonant terms corresponding to the nonzero cross-combinations (7.1). The corresponding formal normal form has two invariant manifolds coinciding with the corresponding coordinate subspaces.

Moreover, all denominators (7.1) are obviously bounded from below. Therefore one can expect that the corresponding transformation converges and the invariant manifolds will exist in the analytic category. This is indeed the case, though the accurate proof is organized along different lines.

Theorem 7.1 (Hadamard–Perron theorem for holomorphic flows)**.** *Assume that the linearization operator of a holomorphic vector field* $Ax + F(x)$ *has a transversal pair of invariant subspaces* $L^{\pm}$ *such that the spectra of* A *restricted on these subspaces are separated from each other.*

Then the vector field has two holomorphic invariant manifolds $W^{\pm}$ *tangent to the subspaces* $L^{\pm}$*.*

However, the proof of this result is indirect. We start by formulating and proving a counterpart of Theorem 7.1 for biholomorphisms.

Definition 7.2. A holomorphic self-map $H \in \operatorname{Diff}(\mathbb{C}^n, 0)$, $x \mapsto Mx + h(x)$, $h(0) = \frac{\partial h}{\partial x}(0) = 0$, is said to be *hyperbolic* if no eigenvalue of the linearization matrix $M \in \operatorname{GL}(n, \mathbb{C})$ has modulus 1.

For a matrix M without eigenvalues on the unit circle, we denote $L^{\pm} \subseteq \mathbb{C}^n$ two invariant subspaces such that the restriction $M|_{L^-}$ is contracting (in a suitable Hermitian metric) and $M|_{L^+}$ expanding (i.e., $M^{-1}|_{L^+}$ is contracting).

To define invariant manifolds for biholomorphisms we need to be careful and replace sets by their germs at the fixed points. Otherwise it would be necessary to give different definitions for expanding and contracting submanifolds.

Definition 7.3. A holomorphic submanifold W passing through a fixed point of a biholomorphism $H\colon (\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$ is *invariant*, if *the germ* of $H(W)$ at the origin coincides with *the germ* of W.

Theorem 7.4 (Hadamard–Perron theorem for biholomorphisms)**.** *A hyperbolic holomorphism in a sufficiently small neighborhood of the fixed point at the origin has two holomorphic invariant submanifolds* W^+ *and* W^-*.*

These manifolds pass through the origin, transversal to each other and are tangent to the corresponding invariant subspaces $L^{\pm}$ *of the linearized map* $x \mapsto Mx$*.*

The dimensions of the invariant manifolds are necessarily equal to the dimension of the corresponding subspaces. The manifold W^+ is called *unstable manifold*, whereas W^- is referred to as the *stable manifold*, because the restriction of H on these manifolds is unstable and stable respectively.

Proof. The linearization matrix M of a holomorphic biholomorphism $H\colon (\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$ can be put into the block diagonal form. Choosing appropriate system of local holomorphic coordinates $(x, y) \in (\mathbb{C}^k, 0) \times (\mathbb{C}^l, 0)$, $k + l = n$, one can always assume that the map H has the form

$$H\colon \begin{pmatrix} x \\ y \end{pmatrix} \longmapsto \begin{pmatrix} Bx + g(x, y) \\ Cy + h(x, y) \end{pmatrix}, \qquad (x, y) \in (\mathbb{C}^k, 0) \times (\mathbb{C}^l, 0). \tag{7.2}$$

Here the square matrices B, C and the nonlinear terms g, h of order $\geqslant 2$ satisfy the conditions

$$\begin{gathered} |B| \leqslant \mu, \quad |C^{-1}| \leqslant \mu, \qquad \mu < 1, \\ |f(x, y)| + |g(x, y)| < |x|^2 + |y|^2, \qquad \text{for } |x| < 1,\ |y| < 1. \end{gathered} \tag{7.3}$$

with some *hyperbolicity parameter* $\mu < 1$.

It is sufficient to prove the existence of the *stable* manifold only; the unstable manifold for H is the stable manifold of the inverse map H^{-1} which is also hyperbolic.

The stable manifold W^+ tangent to $L^+ = \{(x, 0)\}$ is necessarily the graph of a holomorphic vector function $\varphi\colon \{|x| \leqslant \varepsilon\} \to \{|y| \leqslant \varepsilon\}$ defined in a sufficiently small polydisk, $\varphi(0) = 0$, $\frac{\partial \varphi}{\partial x}(0) = 0$. For this manifold to be invariant, the function φ must satisfy the functional equation

$$\varphi\bigl(Bx + g(x, \varphi(x))\bigr) = C\varphi(x) + h(x, \varphi(x)). \tag{7.4}$$

This equation can be transformed to the fixed point form as follows:

$$\varphi = \mathcal{H}\varphi, \qquad (H\varphi)(x) = C^{-1}\varphi\bigl(Bx + g(x, \varphi(x))\bigr) - h(x, \varphi(x)). \tag{7.5}$$

All assertions of Theorem 7.4 follow from the contracting map principle and the following Lemma 7.5. $\square$

The "linearization" (removal of all nonlinear terms of order 2 and higher) of the operator $\mathcal{H}$ at the "point" $\varphi = 0$ results in the operator

$$\varphi(x) \mapsto C^{-1}\varphi(Bx), \qquad |B|,\ |C^{-1}| \leqslant \mu < 1,$$

which is obviously contracting. Lemma 7.5 shows that nonlinear terms do not affect this property.

Denote by $\mathcal{A}_\varepsilon$ the Banach space of functions holomorphic in the open disk of radius $\varepsilon > 0$ and continuous on the closure.

Lemma 7.5. *Under the assumptions* (7.3), *the nonlinear operator* $\mathcal{H}$ *has the following properties*:

(1) *$\mathcal{H}$ is well defined for φ in the ball $\mathcal{B}_\varepsilon = \{\varphi\colon \sup_{|x|<\varepsilon} |\varphi(x)| < \varepsilon\}$ inside the space $\mathcal{A}_\varepsilon$, and takes this ball into itself,*

(2) *the subset $\mathcal{B}^1_\varepsilon$ of functions in $\mathcal{B}_\varepsilon$ with the Lipschitz constant $\leqslant 1$, is preserved by $\mathcal{H}$,*

(3) *the operator $\mathcal{H}$ is contracting on $\mathcal{B}^1_\varepsilon$,*

provided that the value $\varepsilon > 0$ is sufficiently small.

Proof. To prove the first assertion, note that $|Bx + g(x, \varphi(x))| < \mu|x| + |x|^2 + |\varphi|^2 < \mu\varepsilon + 2\varepsilon^2 < \varepsilon$ for $|x| < \varepsilon$, if ε is sufficiently small. Thus the composition occurring in the definition of $\mathcal{H}$ makes perfect sense and $\mathcal{H}\varphi$ is well defined. For the same reason, $|\varphi|$ never exceeds $\mu\varepsilon + 2\varepsilon^2 < \varepsilon$ which means that $\mathcal{B}_\varepsilon$ is taken by $\mathcal{H}$ into itself.

The Jacobian matrix $J(x) = \frac{\partial\varphi}{\partial x}$ is transformed into $J' = C^{-1}J(\cdots)(B+\frac{\partial g}{\partial x}+\frac{\partial g}{\partial y}J)+(\frac{\partial h}{\partial x}+\frac{\partial h}{\partial y}J)$. Since the terms g, h are of order $\geqslant 2$, their derivatives vanish at the origin and therefore the Jacobian is no greater (in the sense of the matrix norm) than $(\mu^2 + O(\varepsilon))|J|$. As $\mu < 1$, this proves the assertion about the Lipschitz constant.

To prove the last assertion that $\mathcal{H}$ is contractive, notice that the operator $\varphi(x) \mapsto h(x, \varphi(x))$ is strongly contracting:

$$|h(x, \varphi_1(x)) - h(x, \varphi_2(x))| \leqslant \left|\tfrac{\partial h}{\partial y}\right| |\varphi_1(x) - \varphi_2(x)| \leqslant O(\varepsilon)\|\varphi_1 - \varphi_2\|_\varepsilon. \quad (7.6)$$

Consider the operator $\varphi \mapsto \mathcal{G}\varphi = \varphi(Bx + g(x, \varphi))$ and the difference of the values it takes on two functions $\varphi_1, \varphi_2 \in \mathcal{B}^1_\varepsilon$: by the triangle inequality,

$$\begin{aligned} |\mathcal{G}\varphi_1(x) - \mathcal{G}\varphi_2(x)| &= |\varphi_1(Bx + g_1(x)) - \varphi_2(Bx + g_2(x))| \\ &\leqslant |\varphi_1(Bx + g_2(x)) - \varphi_2(Bx + g_2(x))| \\ &\quad + |\varphi_1(Bx + g_1(x)) - \varphi_1(Bx + g_2(x))|, \end{aligned}$$

where we denoted $g_i(x) = g(x, \varphi_i(x))$ for brevity. The first term does not exceed $\|\varphi_1 - \varphi_2\|_\varepsilon$. Since the vector function $\varphi_1 \in \mathcal{B}^1_\varepsilon$ has Lipschitz constant 1, the second term does not exceed $|g_1(x) - g_2(x)| = |g(x, \varphi_1(x)) - g(x, \varphi_2(x))|$. Similarly to (7.6), this part is no greater than $O(\varepsilon)\|\varphi_1 - \varphi_2\|_\varepsilon$. Finally, we conclude that $\mathcal{G}$ is Lipschitz on $\mathcal{B}^1_\varepsilon$: $\|\mathcal{G}\varphi_1 - \mathcal{G}\varphi_2\|_\varepsilon \leqslant (1 + O(\varepsilon))\|\varphi_1 - \varphi_2\|$.

Adding all terms together for $\mathcal{H} = C^{-1}\mathcal{G} - h(x, \cdot)$, we conclude that if $\varphi_{1,2} \in \mathcal{B}^1_\varepsilon$, then

$$\|\mathcal{H}\varphi_1 - \mathcal{H}\varphi_2\|_\varepsilon \leqslant (\mu + O(\varepsilon))\, \|\varphi_1 - \varphi_2\|_\varepsilon.$$

Since $\mu < 1$, the operator $\mathcal{H}$ is contracting on the closed subset $\mathcal{B}^1_\varepsilon$ of the complete metric space $\mathcal{B}_\varepsilon \subset \mathcal{A}_\varepsilon$. □

Remark 7.6. Characteristically for the proofs based on the contracting map principle, the germs of invariant manifolds are automatically unique.

Now we can derive Theorem 7.1 from Theorem 7.4.

Proof. Passing if necessary to an orbitally equivalent field, one may assume that the linearization $A = \operatorname{diag}\{A_+, A_-\}$ is block diagonal with the spectra of the blocks are *separated by the imaginary axis.*

Consider the flow maps $\Phi^t = \exp tF\colon (\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$ for $t = 1/k$, $k = 1, 2, \dots$. Each of them is a biholomorphism with the linear part $x \mapsto \exp tAx$ whose eigenvalues are the corresponding exponentials $\{\exp t\lambda_i\colon \lambda_i \in S\}$ separated by the unit circle $\{|\lambda| = 1\}$. In the assumptions of the theorem, each flow map Φ^t is hyperbolic for the specified values of $t \in 1/\mathbb{N}$. By Theorem 7.4, the map Φ^t has a pair of invariant manifolds $W_t^\pm$, tangent to the corresponding invariant subspaces $L^\pm$ common for all $t \in \mathbb{R}$.

Apriori, the invariant subspaces $W_t^\pm$ do not have to coincide. However, $\left(\Phi^{1/k}\right)^k = \Phi^1$, therefore manifolds invariant for $\Phi^{1/k}$, are invariant also for Φ^1. Since the invariant manifolds for the latter map are unique, we conclude that all the maps $\Phi^{1/k}$ leave the pair $W^\pm = W_1^\pm$ invariant.

In other words, an analytic trajectory $x(t)$ of the vector field which begins on, say, W^-, $x(0) \in W^-$, remains on W^- for $t = 1/k$. Since isolated zeros of analytic functions cannot have accumulation points, $x(t)$ is on W^- for all (sufficiently small) values of $t \in (\mathbb{C}, 0)$. Then W^- is invariant for the vector field $Ax + F(x)$. The proof for W^+ is similar. □

Remark 7.7. Intersection of invariant manifolds is again an invariant manifold. This observation allows us to construct small-dimensional invariant manifolds for holomorphic vector fields. For instance, if the linearization matrix Λ has a simple eigenvalue $\lambda_1 \neq 0$ such that $\lambda_1/\lambda_j \notin \mathbb{R}_+$ for all other eigenvalues λ_j, $j = 2, \dots, n$, then the vector field has a *one-dimensional holomorphic invariant manifold* (curve) tangent to the corresponding eigenvector.

The Hadamard–Perron theorem for holomorphic flows, as formulated above, is the nearest analog of the Hadamard–Perron theorem for smooth flows in $\mathbb{R}^n$. There are known stronger results in this direction; see [**Bib79**].

7B. Hyperbolic invariant curves for saddle-nodes. Consider a holomorphic vector field on the plane $(\mathbb{C}^2, 0)$ with the saddle-node at the origin. Recall that by Definition 4.28, this means that exactly one of the eigenvalues is zero, while the other eigenvalue must be nonzero. The null space (line) of the linearization operator is called the *central* direction. The direction of eigenvector with the nonzero eigenvalue is referred to as *hyperbolic.*

The nonzero eigenvalue cannot be separated from the null one, thus the Hadamard–Perron theorem cannot be applied. However, the invariant manifold (smooth holomorphic curve) tangent to the eigenvector with nonzero eigenvalue, exists and is unique in this case as well. As before, we start with the case of biholomorphisms with one contracting eigenvalue $|\mu| < 1$ and the other eigenvalue equal to 1. For obvious reasons, such maps are called *saddle-node* biholomorphisms.

Any saddle-node biholomorphism $H\colon (\mathbb{C}^2, 0) \to (\mathbb{C}^2, 0)$ can be brought into the form

$$H\colon \begin{pmatrix} x \\ y \end{pmatrix} \longmapsto \begin{pmatrix} \mu x + g(x,y) \\ y + y^2 + h(x,y) \end{pmatrix}, \qquad \mu \in (0,1) \subset \mathbb{R}, \tag{7.7}$$

with g, h holomorphic nonlinear terms of order $\geqslant 3$, by a suitable holomorphic choice of coordinates x, y. Indeed, all other quadratic terms are nonresonant and can be removed (Exercise 4.8).

Theorem 7.8. *The biholomorphism* (7.7) *has a unique holomorphic invariant manifold* (*curve*) *tangent to the eigenvector* $(1, 0) \in \mathbb{C}^2$.

Proof. The manifold $W = \operatorname{graph} \varphi$ is invariant for the saddle-node self-map H of the form (7.7) if the function φ satisfies the functional equation

$$\varphi(\mu x + g(x, \varphi(x))) = \varphi(x) + \varphi^2(x) + h(x, \varphi(x)). \tag{7.8}$$

This equation can be represented under the fixed point form $\mathcal{H}\varphi = \varphi$ using the operator $\mathcal{H}$ defined as follows:

$$(\mathcal{H}\varphi)(x) = \varphi\big(\mu x + g(x, \varphi(x))\big) - \varphi^2(x) - h(x, \varphi(x)). \tag{7.9}$$

This operator is no longer contracting: its linearization at $\varphi = 0$ is the operator $\varphi(x) \mapsto \varphi(\mu x)$ which keeps all constants fixed. To restore the contractivity, we have to restrict this operator on the subspace of functions vanishing at the origin, with the norm $\|\varphi\|' = \sup_{x \neq 0} \frac{|\varphi(x)|}{|x|}$. Technically it is more convenient to substitute $\varphi(x) = x\psi(x)$ into the functional equation (7.8) and bring it back to the fixed point form. As a result, we obtain the equation

$$(\mu x + g(x, x\psi(x)) \cdot \psi\big(\mu x + g(x, x\psi(x))\big) = x\psi(x) + x^2\psi^2(x) + h(x, x\psi(x)),$$

which yields the nonlinear operator $\mathcal{H}'$,

$$(\mathcal{H}'\psi)(x) = \big(\mu + g'(x, \psi(x))\big) \cdot \psi\big(\mu x + g(x, x\psi(x))\big) - x\psi^2(x) - h'(x, \psi). \tag{7.10}$$

Here the holomorphic functions $g'(x,y) = g(x,xy)/x$, $h'(x,y) = h(x,xy)/x$ are of order $\geqslant 2$ at the origin.

The proof of Lemma 7.5 carries out almost literally for the operator $\mathcal{H}'$ as in (7.10), proving that it is contractible on the space of functions $\psi\colon \{|x| < \varepsilon\} \to \{|y| < \varepsilon\}$ with respect to the usual supremum-norm on sufficiently small discs. □

Completely similar to derivation of Theorem 7.1 from Theorem 7.4 in the hyperbolic case, Theorem 7.8 implies the following result concerning holomorphic saddle-nodes.

Theorem 7.9. *A holomorphic vector field on the plane* $(\mathbb{C}^2, 0)$ *having a saddle-node singularity* (*one eigenvalue zero, another nonzero*) *at the origin, admits a unique holomorphic nonsingular invariant curve passing through the singular point and tangent to the hyperbolic direction.* □

This curve is called the *hyperbolic invariant manifold*.

It is important to conclude this section by the explicit example showing that the other invariant manifold, the *central manifold* tangent to the central direction, may not exist in the analytic category. Note, however, that the formal invariant manifold always exists and is unique: this follows from the formal orbital classification of saddle-nodes (Proposition 4.29).

Example 7.10 (L. Euler)**.** The vector field

$$x^2\frac{\partial}{\partial x} + (y-x)\frac{\partial}{\partial y} \tag{7.11}$$

has vertical hyperbolic direction $\frac{\partial}{\partial y}$ and the central direction $\frac{\partial}{\partial x} + \frac{\partial}{\partial y}$. The central manifold, if it exists, must be represented as the graph of the function $y = \varphi(x)$, $\varphi(x) = x + \sum_{k\geqslant 2} c_k x^k$. However, this series diverges, as was noticed already by L. Euler. Indeed, the function φ must be the solution to the differential equation

$$\frac{d\varphi}{dx} = \frac{\varphi(x) - x}{x^2}$$

which implies the recurrent formulas for the coefficients,

$$k\,c_k = c_{k+1}, \qquad k = 1, 2, \dots, \qquad c_1 = 1.$$

The factorial series with $c_k = (k-1)!$ has zero radius of convergence, hence no analytic central manifold exists.

However, sufficiently large “pieces” of the central manifold for the saddle-node can be shown to exist; see §22**I**.

Exercises and Problems for §7.

Exercise 7.1. Prove that a nonresonant hyperbolic self-holomorphism is analytically linearizable on its holomorphic invariant manifolds W^+ and W^-.

Problem 7.2. Prove that if a hyperbolic self-map analytically depends on additional parameters (and remains hyperbolic for all values of these parameters), then the invariant manifolds $W^\pm$ also depend analytically on the parameters.

Problem 7.3. Formulate and prove a parallel statement for a saddle-node.

Exercise 7.4. Describe possible number and relative position of analytic separatrices of elementary planar singularities of holomorphic vector fields.

Problem 7.5. Assume that the first k eigenvalues $\lambda_1, \dots, \lambda_k$ from the spectrum of a holomorphic vector field $F \in \mathcal{D}(\mathbb{C}^n, 0)$, $k \leqslant n$, are real, and the others are not.

Prove that the field F has a holomorphic k-dimensional invariant manifold tangent to the coordinate plane generated by the first k basis vectors.

8. Desingularization in the plane

Reasonably complete analysis of singular points of holomorphic vector fields using holomorphic normal forms and transformations by biholomorphisms, is possible under the assumption that the linear part is not very degenerate. The degenerate cases have to be treated by transformations that can alter the linear part. Such transformations, necessarily not holomorphically invertible, are known by the name *desingularization*, *resolution of singularities*, *sigma-process* or *blow-up*. Very roughly, the idea is to consider a holomorphic map $\pi\colon M \to (\mathbb{C}^2, 0)$ of a holomorphic surface (2-dimensional manifold) M that squeezes (blows down) a complex 1-dimensional curve $D \subset M$ to the single point $0 \in \mathbb{C}^2$, while being one-to-one between $M \smallsetminus D$ and $(\mathbb{C}^2, 0) \smallsetminus \{0\}$. The second circumstance allows us to pull back local objects (functions, curves, foliations, 1-forms, *etc.*) from $(\mathbb{C}^2, 0)$ to M and then extend them on D. These pullbacks are called desingularizations, or blow-ups of the initial objects; sometimes M is itself called the blow-up of (the neighborhood of) the point $0 \in \mathbb{C}^2$.

In this section we develop some basic algebraic geometry necessary to deal with desingularizations and introduce the notion of multiplicity of an isolated singularity of a foliation.

Using desingularization one can ultimately simplify singularities of holomorphic foliations in dimension 2. The main result of this section, the fundamental Desingularization Theorem 8.14 asserts that by a suitable blow-up any singular holomorphic foliation in a neighborhood of a singular point can be resolved into a singular foliation, defined in a neighborhood of a union $D = \bigcup_i D_i$ of one or more transversally intersecting holomorphic curves D_i, which has only *elementary* singularities on D.

8A. Polar blow-up. We start with a transcendental but geometrically more transparent construction in the real domain.

Definition 8.1. The *polar blow-down* is the map P of the real cylinder $C = \mathbb{R} \times \mathbb{S}^1 \to \mathbb{R}^2$ onto the plane $\mathbb{R}^2$,

$$P\colon (r, \varphi) \mapsto (r\cos\varphi, r\sin\varphi). \tag{8.1}$$

This map is a real analytic diffeomorphism between the open half-cylinder $C_+ = \{r > 0\} \subset C$ and the punctured plane $\mathbb{R}^2 \smallsetminus \{0\}$. The