Problem 5.5. Let $\mathbf{F} = F(x)x\frac{\partial}{\partial x} \in \mathcal{D}(\mathbb{C}^1, 0)$ be the germ of a holomorphic vector field at a singular point of multiplicity $k+1 \geqslant 2$ at the origin, $F(x) = x^{k+1}(1+o(1))$, and $\mathbf{F}' = \mathbf{F} + \mathbf{o}(x^{2k+1}) \in \mathcal{D}(\mathbb{C}^1, 0)$ is another such germ with the same $2k+1$-jet.

(i) Prove that these two germs are analytically equivalent if and only if two meromorphic 1-forms ω and ω', dual to $\mathbf{F}$ and $\mathbf{F}'$ respectively, are holomorphically equivalent (cf. with Remark 5.27).

(ii) Show that in the assumptions of the problem, the orders of the poles and the Laurent parts of the 1-forms ω and ω' coincide so that the difference $\omega - \omega'$ is holomorphic.

(iii) Passing to the primitives and denoting by $a_k, \dots, a_1, a_0$ the common Laurent coefficients of the forms ω, ω', prove that the equation

$$\frac{a_k}{y^k} + \cdots + \frac{a_1}{y} + a_0 \ln y + O(y) = \frac{a_k}{x^k} + \cdots + \frac{a_1}{x} + a_0 \ln x + O(z)$$

with holomorphic terms $O(y)$ and $O(x)$, admits a holomorphic solution $y = y(x)$ tangent to identity (substitute $y = ux$ and apply the implicit function theorem to the function $u(x)$ with $u(0) = 1$).

Problem 5.6 (Yet another proof of Schröder–Kœnigs theorem; cf. with [**CG93**]). Let $f \in \operatorname{Diff}(\mathbb{C}, 0)$ be a contracting hyperbolic holomorphic self-map, $f(z) = \lambda z + \cdots$, $|\lambda| < 1$, and $g(z) = \lambda z$ its linearization (the normal form).

Prove that the sequence of iterations $h_n = g^{-\circ n} \circ f^{\circ n}$ is defined and converges in some small disk around the origin. The limit $h = \lim h_n$ conjugates f and g.

Problem 5.7. Prove Theorem 5.5 along the same lines (M. Villarini).

6. Finitely generated groups of conformal germs

Thus far we have studied classification and certain dynamic properties of *single* germs of vector fields and biholomorphisms. However, in §2**C** we introduced an important invariant of foliation, the holonomy group of a leaf $L \in \mathcal{F}$ with nontrivial fundamental group $\pi_1(L, a)$, $a \in L$. By construction, the holonomy is a representation of $\pi_1(L, a)$ by conformal germs $\operatorname{Diff}(\tau, a)$, where τ is a cross-section to L at a, and the holonomy group G is identified with the image of that representation. Usually if the fundamental group of a leaf of a holomorphic foliation is finitely generated, then so is the group G. We will consider only the case of holomorphic foliations on complex 2-dimensional surfaces, thus dealing only with finitely generated subgroups of the group $\operatorname{Diff}(\mathbb{C}, 0)$ of conformal germs.

In this section we study classification problems for *finitely generated groups* of conformal germs and their dynamic properties, focusing on the properties which will be later used in §11 and §28. In much more detail the theory is treated in the recent monograph [**Lor99**].

6A. Equivalence of finitely generated groups of conformal germs. The following definition is inspired by Proposition 2.15.

Definition 6.1. Two finitely generated subgroups $G, G' \subseteq \operatorname{Diff}(\mathbb{C}, 0)$ are called analytically (topologically, formally) equivalent if one can choose two systems of generators $G = \langle f_1, \dots, f_n \rangle$ and $G' = \langle f'_1, \dots, f'_n \rangle$ which are simultaneously conjugated by the germ of a holomorphic map (homeomorphism, formal series) h so that $h \circ f_j = f'_j \circ h$ for all $j = 1, \dots, n$.

Remark 6.2. If the generators of two groups are simultaneously conjugated as below, then the groups are isomorphic in the group theoretic sense. Indeed, any relation between generators of one group is automatically true in the second groups and vice versa, since the identical germ $\mathrm{id} \in \operatorname{Diff}(\mathbb{C}, 0)$ can be conjugated only to itself. Thus both groups are isomorphic to the quotients of the free group on n generators by the isomorphic sets of relations.

Example 6.3. Two conformal germs f and g from $\operatorname{Diff}(\mathbb{C}, 0)$ are analytically, topologically or formally equivalent if and only if the cyclic (commutative) subgroups $\{f^{\circ\mathbb{Z}}\}$ and $\{g^{\circ\mathbb{Z}}\}$ of $\operatorname{Diff}(\mathbb{C}, 0)$ generated by these germs are equivalent in the corresponding sense. In particular, they must be both finite or both infinite.

It turns out that some very important information on the analytic structure of the group is encoded in its algebraic properties.

Example 6.4. A generic single conformal germ can be linearized. However, simultaneous linearization (analytic, formal or topological) of two or more germs is possible *only if the group generated by these germs is commutative.* Indeed, the subgroup generated by any finite number of linear germs $f_j : z \mapsto \mu_j z$ in $\operatorname{Diff}(\mathbb{C}, 0)$ is commutative.

The "derivative map"

$$T\colon \operatorname{Diff}(\mathbb{C}, 0) \to \mathbb{C}^*, \qquad Tg = \tfrac{dg}{dz}(0) \in \mathbb{C}^*, \tag{6.1}$$

associating with any germ g its multiplicator at the fixed point at the origin, is a group homomorphism: by the chain rule of differentiation, $T(g \circ f) = Tg \cdot Tf = Tf \cdot Tg$ with the kernel equal to the normal subgroup of germs tangent to the origin, denoted by $\operatorname{Diff}_1(\mathbb{C}, 0)$:

$$\operatorname{Ker} T = \operatorname{Diff}_1(\mathbb{C}, 0) = \{g \in \operatorname{Diff}(\mathbb{C}, 0) \colon g(z) = z + O(z^2)\}. \tag{6.2}$$

Definition 6.5. Elements of the subgroup $\operatorname{Diff}_1(\mathbb{C}, 0)$ tangent to identity, are called *parabolic germs.*

The *parabolic subgroup* $\operatorname{Diff}_1(\mathbb{C},0)$ is *filtered* by the order of contact with the identity:

$$\begin{aligned} \operatorname{Diff}_1(\mathbb{C},0) \smallsetminus \{\mathrm{id}\} &= \mathscr{A}_1 \sqcup \mathscr{A}_2 \sqcup \mathscr{A}_3 \sqcup \cdots, \\ \mathscr{A}_p &= \{g \in \operatorname{Diff}_1(\mathbb{C},0) : g(z) = z\cdot(1 + az^p + \cdots),\ a \neq 0\}. \end{aligned} \tag{6.3}$$

The natural index p in the above formulas will be referred to as the *level* of a conformal germ $g \in \mathscr{A}_p$: this parameter is slightly more convenient to use than the order of tangency between the germ and identity, equal to $p+1$. One can easily verify that the level is invariant (does not change by conjugacy $g \mapsto h\circ g\circ h^{-1}$, $h \in \operatorname{Diff}(\mathbb{C},0)$).

Example 6.6. If the group G has no nontrivial parabolic germs, i.e., $G \cap \operatorname{Diff}_1(\mathbb{C},0) = \{\mathrm{id}\}$, then T is injective and hence G is necessarily commutative as a group isomorphic to a subgroup of the commutative group $\mathbb{C}^*$. Moreover, if G is analytically or formally linearizable, then each element g can be conjugated only with the linear germ $x \mapsto \nu_g x$, $\nu_g = Tg \in \mathbb{C}^*$, since the multiplicators of g and $h\circ g\circ h^{-1}$ necessarily coincide. Yet we wish to stress that being *algebraically* isomorphic to a subgroup of $\mathbb{C}^*$ (e.g., an infinite cyclic subgroup) is not sufficient for linearizability of the group, even on the formal level.

A simple *sufficient* condition for simultaneous linearizability (and hence commutativity) of a finitely generated group is its finiteness.

Theorem 6.7 (Bochner linearization theorem)**.** *Any finite subgroup $G \subseteq \operatorname{Diff}(\mathbb{C},0)$ can be linearized: there exists a biholomorphism $h \in \operatorname{Diff}(\mathbb{C},0)$ such that all germs $h\circ g\circ h^{-1}$ are linear,*

$$\forall g \in G \qquad h\circ g\circ h^{-1}(x) = \nu_g x, \qquad \nu_g = Tg \in \mathbb{C}^*. \tag{6.4}$$

Proof. Define the germ of the analytic *function* $h \in \mathcal{O}(\mathbb{C},0)$ by the formula

$$h = \sum_{g\in G} (Tg)^{-1}\cdot g$$

in any chart on $(\mathbb{C},0)$ (note that the addition makes sense only in $\mathcal{O}(\mathbb{C},0)$, but not in $\operatorname{Diff}(\mathbb{C},0)$). The germ h has the linear part $Th = \sum_g 1 = |G| \neq 0$ and is therefore invertible.

By the chain rule T, for any germ $f \in G$ we have

$$\begin{aligned} h\circ f = \sum_{g\in G} (Tg)^{-1}\cdot(g\circ f) = Tf\cdot \sum_{g\in G} \bigl(T(g\circ f)\bigr)^{-1}\cdot(g\circ f) \\ = Tf\cdot \sum_{g'\in G} (Tg')^{-1}\cdot g' = Tf\cdot h, \end{aligned}$$

which means that h conjugates f with the multiplication by $\nu_f = Tf$. □

This linearization theorem implies a simple but useful corollary. Recall that for nonhyperbolic germs with multiplicators on the unit circle the problem of convergence of linearizing transformations is in general very difficult for the nonresonant case; see §5**E**. The resonant case turns out to be unexpectedly simple.

Theorem 6.8. *A* resonant *conformal germ* $f: z \mapsto \mu z + \cdots \in \operatorname{Diff}(\mathbb{C}, 0)$ *with* $\mu \in \exp 2\pi i\mathbb{Q}$, *is formally linearizable if and only if it is analytically linearizable.*

Proof. Only one direction of the equivalence is nontrivial. Assume that h is a formal germ linearizing the germ f. Since the multiplicator μ is a root of unity, $(h \circ f \circ h^{-1})^{\circ n} = h \circ f^{\circ n} \circ h^{-1} = \text{id}$ for some finite order n. This means that the formal series h conjugates the holomorphic germ $f^{\circ n}$ with the identity. Yet the only *holomorphic map formally equivalent to identity* is the identity itself, hence $f^{\circ n} = \text{id}$ and thus f is periodic (generates a finite group). By Theorem 6.7, f is analytically linearizable. □

One can replace finiteness of the group in the Linearization Theorem 6.7 by the assumption that *all elements of this group have finite order.*

Theorem 6.9. *A finitely generated subgroup of germs* $G \subset \operatorname{Diff}(\mathbb{C}, 0)$ *whose elements all have finite order, is analytically linearizable and finite, hence commutative and cyclic.*

Proof of Theorem 6.9. If the group is noncommutative, then it contains an element $\text{id} \neq f \in \operatorname{Diff}_1(\mathbb{C}, 0)$ (cf. with Example 6.6). Such an element always has an infinite order in contradiction with our assumptions: if $f(z) = z + cz^{p+1} + \cdots$, $c \neq 0$, then $f^n(z) = z + nc\, z^{p+1} + \cdots \neq \text{id}$. Thus G must be commutative.

A commutative group generated by finitely many elements of finite orders, is itself finite. By Theorem 6.7, the group G is analytically conjugate to a finite multiplicative subgroup of $\mathbb{C}^*$. All such subgroups are cyclic and generated by appropriate primitive roots of unity. □

6B. First steps of formal classification. In this subsection we study *formal* classification of finitely generated groups of conformal germs.

6**B**$_1$. *Solvable and metabelian groups.* Recall that the commutator $[G, G]$ of an (abstract) group G is the group generated by all commutators of pairs of elements $[f, g] = f \circ g \circ f^{-1} \circ g^{-1}$; it is a subgroup in G. Moreover, since $T[f, g] = 1$, the commutator $[G, G]$ is a subgroup in $\operatorname{Diff}_1(\mathbb{C}, 0)$.

A group is *solvable*, if the decreasing chain of iterated commutators stabilizes on the trivial group:

$$\begin{gathered} G^0 \supseteq G^1 \supseteq G^2 \supseteq \cdots \supseteq G^{\ell-1} \supsetneqq G^\ell = \{\mathrm{id}\}, \\ G^0 = G, \qquad G^{k+1} = [G^k, G^k], \ k = 0, 1, 2, \dots . \end{gathered} \tag{6.5}$$

If G is commutative (abelian), then $\ell = 1$. Solvable groups with $\ell = 2$ are called *metabelian*: their first commutators are commutative.

While for arbitrary groups the index ℓ may take any finite value, for subgroups of $\mathrm{Diff}(\mathbb{C}, 0)$ the only possibilities are $\ell = 0, 1$ (abelian and metabelian respectively) or $\ell = \infty$ (for nonsolvable groups). In other words, we have the following alternative.

Theorem 6.10 (Tits alternative for groups of conformal germs)**.** *A finitely generated subgroup $G \subset \mathrm{Diff}(\mathbb{C}, 0)$ is either metabelian (commutative or noncommutative), or nonsolvable.*

To prove this result, we start with a simple computation (in part explaining, why the level is more convenient to deal with than the order of tangency with the identity).

Proposition 6.11. *For two germs of different levels $p \neq q$ their commutator has the level $p + q$. More specifically, if $f(z) = z + az^{p+1} + \cdots$, $g(z) = z + bz^{q+1} + \cdots$ with $p, q > 0$, then*

$$[f, g](z) = z + ab(p - q)z^{p+q+1} + \cdots . \tag{6.6}$$

Proof. The identity (6.6) is an assertion on the leading term of the germ of the *function* $\left(f \circ g \circ f^{-1} \circ g^{-1}\right)(z) - z \in \mathcal{O}(\mathbb{C}, 0)$ in any holomoprphic chart z. This leading term is not changed if we change the local coordinate from z to $t = f^{-1} \circ g^{-1}(z)$. In the new chart $z = (g \circ f)(t)$ and the leading term of the difference $(f \circ g)(t) - (g \circ f)(t)$ can be computed directly:

$$\begin{aligned} (f \circ g)(t) - (g \circ f)(t) &= t(1 + bt^q + \cdots)(1 + at^p(1 + bt^q + \cdots)^p + \cdots) \\ &\quad - t(1 + at^p + \cdots)(1 + bt^q(1 + at^p + \cdots)^q + \cdots) \\ &= ba\, pt^{p+q+1} - ab\, qt^{q+p+1} + \cdots . \end{aligned}$$

This proves (6.6). □

Remark 6.12. A similar (even easier) computation with $q = 0$ yields the following: if $g(z) = bz + \cdots$, $b \neq 1$, and f as above, then

$$[f, g](z) = z + a(b^p - 1)z^{p+1} + \cdots . \tag{6.7}$$

This computation immediately implies the following alternative for groups of conformal germs tangent to identity.

Lemma 6.13. *A finitely generated subgroup G of* $\mathrm{Diff}_1(\mathbb{C},0)$ *is either commutative or nonsolvable.*

Proof. If $G = G^0$ contains two germs of different *positive* levels $p \neq q$, $p, q > 0$ then it also contains the germ of level $p+q$ (again different from both p and q). Proceeding this way, we construct infinitely many germs of different levels, all belonging to the commutator $G_1 = [G, G]$. Thus G^1 also contains at least two germs of different levels which allows us to conclude that all iterated commutators $G^k = [G^{k-1}, G^{k-1}]$ are nontrivial.

If all germs in G are of the same level $p \geqslant 1$, then the group is in fact commutative. Indeed, in this case the commutator of any two germs $f, g \in G$, if nontrivial, must have the level *strictly greater* than p (again by (6.6)), which again leads to nonsolvability. Hence $[f, g]$ should be identity and the group G commutative. □

Theorem 6.10 is now one step away.

Proof of Theorem 6.10. For *any* group $G \subseteq \mathrm{Diff}(\mathbb{C},0)$ its commutator $G^1 = [G, G]$ belongs to $\mathrm{Diff}_1(\mathbb{C},0) = \ker T$ and therefore can be either trivial (and then G is commutative) or commutative (and then G is metabelian noncommutative) or nonsolvable (and then G is also nonsolvable), by Lemma 6.13. □

Remark 6.14. The same argument shows that if G is a subgroup of $\mathrm{Diff}(\mathbb{C},0)$ *disjoint* from $\mathrm{Diff}_1(\mathbb{C},0)$ (apart from the identical germ), then G is necessarily commutative, as $[G, G] \subseteq G \cap \mathrm{Diff}_1(\mathbb{C},0) = \{\mathrm{id}\}$; cf. with Example 6.6.

6$\mathbf{B}_2$. *Centralizers and symmetries.* Solvable subgroups admit a rather accurate classification on the level of formal equivalence: unless formally linearizable, they are all formally equivalent to subgroups of (twisted) flows of certain nonhyperbolic vector fields. To establish this fact, we need a description of symmetries of parabolic germs.

A *centralizer* of an element g in a group G is the set $Z(g) \subseteq G$ of all elements $f \in G$ commuting with g: $Z(g) = \{f \in G \colon [f, g] = 0\}$. One can instantly verify that the centralizer is a subgroup of G, but in general this subgroup does not have to be commutative.

A parallel notion for the vector fields is a *symmetry*: a germ $g \in \mathrm{Diff}(\mathbb{C},0)$ is called a symmetry of a vector field $F \in \mathcal{D}(\mathbb{C},0)$ (interpreted as a derivation $\mathbf{F}$ of the algebra $\mathcal{O}(\mathbb{C},0)$), if $g^*\mathbf{F} = \mathbf{F}g^*$, in other words, if g transforms F into itself. We will (lacking a better term) call $g \in \mathrm{Diff}(\mathbb{C},0)$ an *orbital symmetry* of a vector field $F \in \mathcal{D}(\mathbb{C},0)$, if g conjugates F with its constant multiple λF, $\lambda \in \mathbb{C}^*$. The construction is identical in the formal context (i.e., for operators on the ring $\mathbb{C}[[z]]$).

If g is a symmetry of F, then g commutes with any flow map $f^t = \exp tF$. In general, mere commutativity of g and $f = \exp F$ is not sufficient for g to be a symmetry of F. Nevertheless, if f is parabolic, the inverse holds.

Recall (Theorem 3.17) that any parabolic germ $f \in \operatorname{Diff}_1(\mathbb{C}, 0)$ is *formally* embeddable: there exists a formal vector field $F \in \mathcal{D}[[\mathbb{C}, 0]]$ such that $f = \exp F$. Without loss of generality we may assume that F is brought to the formal normal form,

$$F = F_{p,a} = z^{p+1}(1 + az^p)\tfrac{\partial}{\partial z}, \qquad a \in \mathbb{C}, \quad p \in \mathbb{N}, \tag{6.8}$$

where p is equal to the level of f (Theorem 4.24).

Lemma 6.15. *If $g \in \operatorname{Diff}(\mathbb{C}, 0)$ is a symmetry of a parabolic germ or a formal series $f = \exp F \in \operatorname{Diff}_1(\mathbb{C}, 0)$, then g is also the symmetry of the field F.*

Proof. Let $\mathfrak{A}$ be the algebra of analytic germs $\mathcal{O}(\mathbb{C}, 0)$ or formal series $\mathbb{C}[[z]]$ respectively (depending on the context).

Consider the operators (automorphisms) $\mathbf{g}, \mathbf{f} \in \operatorname{Aut}\mathfrak{A}$, corresponding to the self-maps g and f, and denote by $\mathbf{F} \in \operatorname{Der}\mathfrak{A}$ the derivation corresponding to the field $F \in \mathcal{D}(\mathbb{C}, 0)$. If g is a symmetry of f, then $\mathbf{g}$ commutes with $\mathbf{f}$.

The derivation $\mathbf{F}$ can be restored from the isomorphism $\mathbf{f}$ by the formal logarithmic series (3.12),

$$\mathbf{F} = (\mathbf{f} - \mathrm{id}) - \frac{1}{2}(\mathbf{f} - \mathrm{id})^2 + \frac{1}{3}(\mathbf{f} - \mathrm{id})^3 \mp \cdots,$$

which stabilizes on the level of any finite order jets, since the difference $\mathbf{f} - \mathrm{id}$ is nilpotent; cf. with Theorem 3.14.

If $\mathbf{g}$ commutes with $\mathbf{f}$, then by the above identity $\mathbf{g}$ commutes also with $\mathbf{F}$, that is, the self-map g is a symmetry of the corresponding vector field F. □

Symmetries (generalized and orbital) of a nonhyperbolic vector field can be easily described. Without loss of generality we can consider only vector fields in the polynomial normal form (6.8).

Proposition 6.16. *A symmetry group of a vector field $F = F_{p,a}$ is the subgroup $G_{p,a} \subset \operatorname{Diff}(\mathbb{C}, 0)$ of the form*

$$G_{p,a} = \{b \cdot \exp tF_{p,a} \colon b \in \mathbb{C}^*,\ b^p = 1,\ t \in \mathbb{C}\} \cong \mathbb{Z}_p \times \mathbb{C}. \tag{6.9}$$

A nontrivial orbital symmetry g with $\lambda \neq 1$ may exist only if $a = 0$ (i.e., if the field is homogeneous), and then the orbital symmetry group is *the semi-direct product,*

$$G'_{p,0} = \{b \cdot \exp F_{p,0} \colon b \in \mathbb{C}^*,\ t \in \mathbb{C}\} \cong \mathbb{C}^* \rtimes \mathbb{C}. \tag{6.10}$$

Note that the groups $G_{p,a}$ and $G'_{p,0}$ indeed consist of symmetries (resp., orbital symmetries) of the field $F_{p,a}$ in the normal form (6.8). Thus the description given by Proposition 6.16, is exact.

Corollary 6.17. *The centralizer $Z(f)$ of a parabolic element $f \in \operatorname{Diff}_1(\mathbb{C}, 0)$ of level p in the group $\operatorname{Diff}(\mathbb{C}, 0)$ is formally equivalent to the group $G_{p,a} \cong \mathbb{Z}_p \times \mathbb{C}$ of germs of the form* (6.9).

Proof. This follows from Proposition 6.16 and Lemma 6.15. □

Corollary 6.18. *The centralizer of any parabolic element $f \in \operatorname{Diff}(\mathbb{C}, 0)$ is a commutative subgroup in $\operatorname{Diff}(\mathbb{C}, 0)$.* □

Remark 6.19. The orbital symmetry group $G'_{p,0}$ is solvable but nonabelian: the composition law for this group has the form

$$(b, t) \circ (b', t') = (bb', tb'^{-p} + t') \neq (b', t') \circ (b, t). \tag{6.11}$$

Yet the commutator $[G'_{p,0}, G'_{p,0}]$ consists of all flow maps and hence is commutative.

Proof of Proposition 6.16. Instead of the polynomial normal form (6.8), we will use the rational normal form

$$F'_{p,a} = \frac{z^{p+1}}{1 - az^p} \cdot \frac{\partial}{\partial z} \tag{6.12}$$

with the same $p \in \mathbb{N}$ and $a \in \mathbb{C}$: the fields $F_{p,a}$ and $F'_{p,a}$ are analytically equivalent; see Remark 4.25.

Let $g \in \operatorname{Diff}(\mathbb{C}, 0)$ be an analytic germ, given in some chart z by the germ of the function $w = g(z) \in \mathcal{O}(\mathbb{C}, 0)$. This germ will be an orbital symmetry of $F'_{p,a}$ if and only if the function $w(z)$ satisfies the ordinary differential equation

$$\frac{dw}{dz} \cdot \frac{z^{p+1}}{1 - az^p} = \lambda \cdot \frac{w^{p+1}}{1 - aw^p}. \tag{6.13}$$

This differential equation has separating variables and can be immediately integrated by reducing it to the Pfaffian form:

$$\frac{(1 - az^p)\, dz}{z^{p+1}} = \lambda \cdot \frac{(1 - aw^p)\, dw}{w^{p+1}}.$$

Note that the equality between two meromorphic 1-forms is possible only if their residues at the origin, equal to $-a$ and $-\lambda a$ respectively, coincide. Thus a nontrivial ($\lambda \neq 1$) orbital symmetry is possible only for a homogeneous vector field (with $a = 0$).

To find all genuine symmetries (with $\lambda = 1$), we integrate the above identity and obtain the equality

$$\frac{1}{pz^p} + a\ln z = \frac{1}{pw^p} + a\ln w - t, \tag{6.14}$$

where $t \in \mathbb{C}$ is a constant of integration. Replacing the germ g by another germ $g \circ (\exp tF)$, we can without loss of generality assume that the constant of integration is equal to zero, $t = 0$. Since the germ g is analytic, the solution w can be represented under the form $w(z) = z\,u(z)$, with an analytic nonvanishing function $u(\cdot)$. Substituting this ansatz into the above formula, we arrive at the identity

$$\frac{1}{pz^p}(1 - u(z)^{-p}) = a\ln u(z), \qquad u(0) \neq 0.$$

The right hand side is holomorphic at the origin, whereas the left hand side has a pole unless $u^p \equiv 1$, i.e., $u(z) \equiv b$ is a constant (root of unity). Then we necessarily have $a = 0$ and the map g must be linear (modulo a flow map, as mentioned above). □

6$\mathbf{B}_3$. *Formal classification of solvable subgroups.* The formal classification of *cyclical* abelian groups coincides with that of their generators and was given in §4**I** (Theorem 4.26). The first nontrivial classification problem concerns *noncyclical* abelian groups.

Theorem 6.20. *A commutative group G which contains no nontrivial parabolic germs, is formally linearizable, i.e., formally equivalent to a subgroup of linear maps $\mathbb{C}^* \subset \operatorname{Diff}(\mathbb{C}, 0)$.*

Proof. If G contains a germ with a nonresonant multiplicator $\mu \notin \exp 2\pi\mathbb{Q}$, then such a germ is formally linearizable. By Remark 6.14, the group must be commutative, yet any germ commuting with the linear map $z \mapsto \mu z$ is itself linear, as it follows immediately from (6.7).

Thus the only remaining possibility is that $TG \subseteq \exp 2\pi i\mathbb{Q}$. But all such germs must be periodic, since their appropriate iteration powers must be parabolic. By Theorem 6.9, this group is analytically linearizable. □

We note that the multiplicative group $\mathbb{C}^*$ can be described in a way similar to (6.9) as the flow group of any *hyperbolic* germ of vector field, e.g., $F(z) = z$,

$$\mathbb{C}^* = \{g(z) = (\exp t) \cdot z \colon t \in \mathbb{C}\} \subset \operatorname{Diff}(\mathbb{C}, 0). \tag{6.15}$$

Theorem 6.21 (classification of abelian nonlinearizable groups)**.** *If a finitely generated group G is commutative and contains a nontrivial parabolic element of some level p, then G is formally equivalent to a subgroup of the group $G_{p,a} \cong \mathbb{Z}_p \times \mathbb{C}$ as in (6.9) for some complex $a \in \mathbb{C}$.*

Proof of the theorem. Because of the commutativity of the group G, it must belong to the centralizer (in $\operatorname{Diff}(\mathbb{C},0)$) of its nontrivial parabolic element f which is described in Corollary 6.17. □

Theorem 6.22 (classification of noncommutative metabelian groups)**.** *Any metabelian noncommutative group G is formally equivalent to a subgroup of the group $G'_{p,0}$ for some finite level p.*

Proof. 1. The parabolic subgroup $G_1 = G \cap \operatorname{Diff}_1(\mathbb{C},0)$ must be commutative by Lemma 6.13 and nontrivial by Remark 6.14. Therefore G_1 belongs to the centralizer (in $\operatorname{Diff}_1(\mathbb{C},0)$) of any its nontrivial element $f \in G_1$ and hence is formally equivalent to a subgroup of $\exp(\mathbb{C}F) = \{\exp tF : t \in \mathbb{C}\}$. Without loss of generality we assume from the very beginning that $G_1 \subseteq \exp(\mathbb{C}F)$, where F is a vector field in the formal normal form (6.8).

2. Since G is noncommutative, there exists another element $h \in G$ not commuting with f. Indeed, the centralizer of f in the bigger group $\operatorname{Diff}(\mathbb{C},0)$ is still *commutative* by Corollary 6.18. Since G is noncommutative, $G \smallsetminus Z(f) \neq \varnothing$.

3. The subgroup $G_1 = G \cap \operatorname{Diff}_1(\mathbb{C},0)$ of parabolic elements of G is a normal subgroup, hence $h \circ G_1 \circ h^{-1} \subseteq G_1 \subseteq \exp(\mathbb{C}F)$. Thus we conclude that $f' = h \circ f \circ h^{-1} = \exp \lambda F$; by our choice of h, the constant λ is different from 1. In other words, h is a nontrivial orbital symmetry of the field F.

By the second assertion of Proposition 6.16, F must be homogeneous and h must belong to the subgroup $G'_{p,0}$ as in (6.10).

4. Any other element $h' \in G$ may either commute with f or not. In the first case by Corollary 6.17 we conclude that $h' \in G_{p,0} \subsetneqq G'_{p,0}$. In the second case $h' \in G'_{p,0}$ by the arguments of step 3 above. □

Remark 6.23. From the proof of Theorem 6.22 it immediately follows that a metabelian noncommutative group is *analytically* equivalent to a subgroup of $\mathbb{C} \cdot \exp(\mathbb{C}F_{p,0})$ for some p, if at least one parabolic germ from G is analytically embeddable.

6C. Integrable germs. Finitely generated groups may possess certain symmetry. Because of the intimate connections with the geometry of foliations, such groups are called *integrable*.

Definition 6.24. A *symmetry group* of the germ of an analytic function $u \in \mathcal{O}(\mathbb{C},0)$ is the subgroup $S_u = \{g \in \operatorname{Diff}(\mathbb{C},0) : u \circ g = u\}$ of holomorphisms preserving u.

Conversely, we say that an analytic germ u is the *first integral* of a group $G \subseteq \operatorname{Diff}(\mathbb{C},0)$, if $G \subseteq S_u$. The group G is said then to be *integrable*.

If G is cyclic and generated by a holomorphism g, then we say that u is a *first integral* of g. The germ g is *integrable* if it admits a nontrivial holomorphic first integral.

Proposition 6.25. *An holomorphism is periodic if and only if it is integrable.*

More precisely, $h \in \operatorname{Diff}(\mathbb{C}, 0)$ *admits a first integral* $u(z) = cz^m + \cdots$, $c \neq 0$, *if and only if* $h^k = \mathrm{id}$, *where* k *divides* m.

Proof. A periodic holomorphism h is linearizable by Theorem 6.7 and any linear map $x \mapsto \nu x$, $\nu^k = 1$, has the first integrals $u(z) = z^m$ for all m divisible by k (the case $m = 0$ is trivial and has to be excluded).

Conversely, if h is integrable and $u(z) = z^m + \cdots$ is the integral, then every level set $M_c = \{u(z) = c\} \subseteq (\mathbb{C}, 0)$ in a sufficiently small neighborhood of 0 consists of exactly m points that are permuted by h. By the Lagrange theorem, $h|_{M_c}$ is of period $k = k(c)$ that divides m. Let k be the minimal value such that the set of k-periodic points is infinite. Then the kth iterate of h is identity by the uniqueness theorem. $\square$

From this proposition and Theorem 6.9 we immediately derive the following necessary condition of integrability.

Corollary 6.26. *An integrable group is finite cyclic (commutative).* $\square$

Remark 6.27. Any germ of a holomorphic function $u(z) = cz^m + \cdots$ of finite order m admits a cyclic symmetry group of order m. The group is generated by the germ of a self-map f which is the linear rotation by the primitive root of unity of order m in the holomorphic chart $w = z \cdot (c + \cdots)^{1/m}$, in which the function itself becomes a monomial.

* * *

Thus far we concentrated on commutative (finite or infinite) and metabelian groups, which are relatively tame. As was already shown, they admit simple formal classification based on the formal type of a single nontrivial parabolic element from the group. Topological classification of solvable groups is also relatively simple and can be derived from Theorem 21.2 (see §21) which claims that the only topological invariant of a parabolic germ is its level. The analytic classification of solvable groups of germs can be reduced to that of the nontrivial parabolic element as above. The corresponding analytic theory is developed in §21 and involves *nonpolynomial normal forms*; see Chapter IV. In summary,

(1) dynamics of solvable groups is relatively simple, in particular,

(2) they have no limit cycles, and

(3) their analytic classification is much finer than the formal one, and the latter in turn is finer than topological classification.

For nonsolvable groups all of these properties fail. In the remaining part of this section we will show that a *generic* (nonsolvable) finitely generated group:

(1) has dense orbits, among which

(2) there exist countably many (properly defined) complex limit cycles. Moreover,

(3) generic groups are *rigid*: two such groups can be topologically equivalent if and only if they are analytically equivalent.

These phenomena will again manifest themselves for singular holomorphic foliations on $\mathbb{P}^2$: the subject will be treated in detail in §28.

The term *generic* in application to finitely generated pseudogroups will mean the following. We fix the number n (usually 2 or more) of generating germs and say that a certain property is *generic*, if it holds for all n-tuples of germs whose jets of some finite order r belong to a "massive" (say, open dense or full measure) subset of the total jet space $\bigoplus_{n \text{ times}} J^r(\mathbb{C}, 0)$.

Example 6.28. A generic group with $n \geqslant 2$ generators is noncommutative and, moreover, nonsolvable.

Indeed, both generators generically are hyperbolic (their multiplicators are off the unit circle). Since the above definition of genericity does not depend on the choice of the chart, without loss of generality, we may assume that one of the generators, f_1, is linear hyperbolic. The group will be noncommutative if the second generator f_2 in this chart is nonlinear (the second Taylor coefficient of f_2 is nonzero).

The commutator $h = [f_1, f_2]$ will be a parabolic element which is generically of level 1 (i.e., tangent to identity with a quadratic nonlinearity). Another parabolic element of level 1 is the commutator $[f_1, h]$. One can show that generically $[[f_1, h], h]$ will be nonzero and hence, by (6.6), have level 2 or more, which would imply nonsolvability.

Usually we will omit routine checks that a certain collection of requirements is fulfilled for a generic finite generated group: in more details various properties determined by finite or infinite order jets, will be discussed in §10, where the notion of *decidable* properties will be introduced.

6D. Dynamics generated by finitely generated groups of germs: pseudogroups. We need first to introduce a proper language for describing *dynamical properties* of finitely generated groups of conformal germs.