§2 Method of Abelian Integrals

6.13. Perturbations of Hamiltonian systems. These are the systems V_ϵ of the type

$$\dot{x} = \frac{\partial H}{\partial y} + \epsilon P(x, y; \epsilon), \ \dot{y} = -\frac{\partial H}{\partial y} + \epsilon Q(x, y; \epsilon),$$

where ϵ is a small parameter.
Before perturbation, i.e. for $\epsilon = 0$, the phase space contains domains filled completely with the *ovals* of levels of the Hamilton function , i.e. the connected components of the (real) curves $\{H(x, y) = h\}$.
After perturbation usually there remain only a finite number of closed phase curves. The problem is to count their number. More precisely, if γ_ϵ is a limit cycle for V_ϵ such that $\gamma_\epsilon \to \gamma(h_i) \subset \{H = h_i\}$, then we say that the oval $H = h_i$ generates a limit cycle. We calculate the number of such h_i's.
This problem can be treated as the linearization of the Hilbert XVI-th problem in a Hamiltonian vector field.
Such questions appear naturally in the bifurcation theory as the below examples show.

6.14. Example (Bogdanov–Takens bifurcation). This is the following 2-parameter bifurcation corresponding to the nilpotent Jordan cell with zero eigenvalues

$$\dot{x} = y, \quad \dot{y} = -\mu_1 + \mu_2 y + x^2 + xy.$$

Here the singular points are $y = 0$, $x_{1,2} = \pm\sqrt{\mu_1}$, $\mu_1 \geq 0$ with the linear parts $\begin{pmatrix} 0 & 1 \\ 2x & \mu_2 \end{pmatrix}$.
We see that the line $\mu_1 = 0$ is bifurcational, with the saddle–node bifurcation (two singular points disappear).
Also the line $\mu_2 = 0$ ($Tr = 0$) is bifurcational, with the Andronov–Hopf bifurcation. Thus, after passing through this line in the direction of growing μ_2, a limit cycle is born (see Figure 6). The problem is what happens with this cycle, when the point in the parameter space is away from the line $\mu_2 = 0$. It turns out that the system can be reduced to a perturbation of a Hamiltonian system.
If the terms $\mu_2 y$ and xy are negligibly small with respect to μ_1, then the system is Hamiltonian with the Hamilton function $\frac{1}{2}y^2 + \mu_1 x - \frac{1}{3}x^3$. Here $x \sim \sqrt{\mu_1}$, $y \sim \mu_1^{3/4}$.
This implies $xy \sim \mu_1^{5/4} << \mu_1$ and $\mu_2 y << \mu_1$ for $\mu_2 << \mu_1^{1/4}$.
After normalization $x = \mu_1^{1/2} X$, $y = \mu_1^{3/4} Y$ and division of the vector field by $\mu_1^{1/4}$ we get

$$\dot{X} = Y, \quad \dot{Y} = -1 + X^2 + \epsilon(\nu + X)Y,$$

where $\epsilon = \mu_1^{1/4}$, $\nu = \mu_2/\sqrt{\mu_1}$.
This bifurcation was first completely investigated by Bogdanov in [**Bog**] who proved that this vector field has at most one limit cycle and its bifurcations are as in Figure 6 (see also [**Arn5**]).

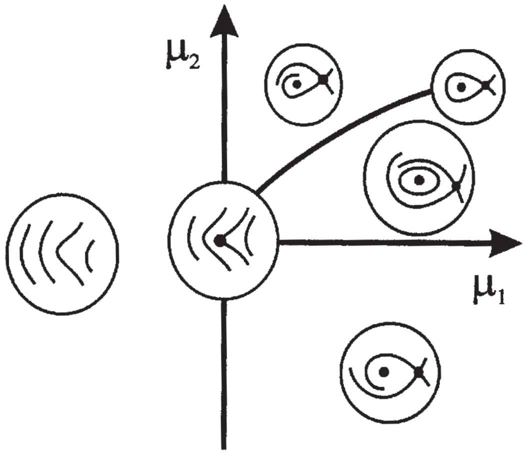

Figure 6

6.15. Example (A resonant periodic trajectory in space). Let $\gamma \subset \mathbb{R}^3$ be a closed trajectory of an unperturbed vector field V_0. Assume additionally that the eigenvalues of the linearization of the Poincaré map lie in the unit circle $\lambda_{1,2} = e^{2\pi i \alpha}$. If α is irrational, then we have a singularity of codimension 1 (in principle). In the resonant case $\alpha = p/q$ one should consider a 2-parameter deformation of this situation.

The section S transversal to γ can be parameterized by points from the complex plane $z \in \mathbb{C}$. We choose such a neighborhood of γ, parameterized by ($\varphi \pmod{2\pi}$, z), that the linear parts of the natural correspondence maps $\{\varphi = \varphi_1\} \to \{\varphi = \varphi_2\}$ (defined by trajectories of V_0) are the homogeneous rotations $z \to e^{i(\varphi_2 - \varphi_1)p/q} z$. This system of linear maps defines the *Seifert foliation* near γ. Its generic leaf makes q turns along γ before closing-up.

Now we take the deformed vector field V_μ and average its z-component along the leaves of the Seifert foliation, i.e. we take $\int_0^{2\pi q} \dot{z}$. We obtain a planar vector field, which is invariant with respect to the rotation by the angle $2\pi/q$ and whose dynamics gives a rather good approximation of the dynamics of V_μ.

The versal families of such invariant vector fields are given in the following formulas (see [**Arn5**]):

$$\dot{x} = y, \ \dot{y} = -\mu_1 + \mu_2 y + x^2 + xy, \quad (q = 1),$$

$$\dot{x} = y, \ \dot{y} = \mu_1 x + \mu_2 y + a x^3 + b x^2 y, \quad (q = 2),$$

$$\dot{z} = \mu z + A|z|^2 + B\bar{z}^{q-1}, \quad \mu = \mu_1 + i\mu_2, \quad (q \geq 3).$$

We see that the case $q = 1$ is the Bogdanov–Takens bifurcation. If $q = 2$, then for $\mu_2 = b = 0$ the system is Hamiltonian with the Hamilton function $(2y^2 - 2\mu_1 x^2 - ax^4)/4$. If $q \geq 3$, then for $\mu_1 = \operatorname{Re} A = 0$ the system is Hamiltonian. It follows from the formula $\operatorname{div} P = 2 \operatorname{Re} \partial P/\partial z$ for a vector field $\dot{z} = P(z, \bar{z})$.

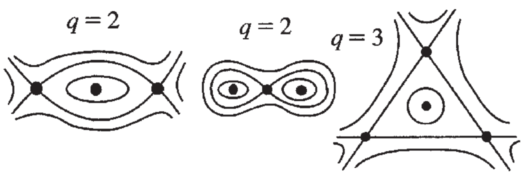

Figure 7

In the cases $q = 2, 3$ the analysis of phase portraits is reduced to analysis of limit cycles in perturbation of the Hamiltonian system; it was done by E. I. Horozov [**Hor**] and by Yu. S. Il'yashenko [**Il1**]. The cases $q \geq 5$ are called *weak resonances* and are simple to investigate (see [**Arn5**]).
The case $q = 4$ is still not finished. The Abelian integrals were studied by A. I. Neishtadt in [**Nei**], by F. S. Berezovskaya and A. I. Khibnik [**BKh**] and by B. Krauskopf [**Kra**].

6.16. Example (One zero and a pair of imaginary eigenvalues). Assume that an unperturbed system in $\mathbb{R}^3$ has a singular point with these eigenvalues of the linear part. It is a codimension 2 phenomenon.
Here one performs the averaging along the trajectories of the linear system (circles) and obtains the following 2-dimensional vector field, where one variable is the amplitude of oscillations):

$$\dot{x} = \mu_1 + \mu_2 x + ax^2 \pm y^2 + by^3, \ \dot{y} = -2xy.$$

If $\mu_2 = b = 0$, then the system has a center. It is not Hamiltonian but it has the first integral

$$y^a \left(x^2 \pm y^2/(a+2) + \mu_1/a\right).$$

This bifurcation was analyzed in [**Zo1**] (see also [**KoZe**]).

6.17. Example (Two pairs of imaginary eigenvalues). This case, after averaging along the 2-tori corresponding to the two independent rotations (of the linear part), gives rise to the generalized Lotka–Volterra system

$$\dot{x} = x(\mu_1 + ax + by), \quad \dot{y} = y(\mu_2 + cx + dy + ex^2).$$

Here also we obtain a situation with perturbation of a system with the first integral

$$x^\alpha y^\beta (1 + kx + ly).$$

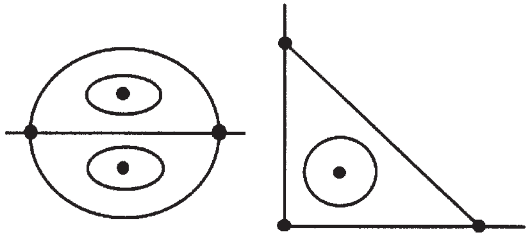

Figure 8

The bifurcations and corresponding Abelian integrals were studied in [**Zo2**] (see also [**KoZe**]).

6.18. Reduction to zeroes of Abelian integrals. Assume that we have the situation as in 6.13. Take a section (interval) S transversal to the family of closed curves $H(x, y) = t$. We parameterize it by the function H restricted to it, $t = H|_S$.
Beginning from here we denote the values of the Hamilton function by t (not by h). This notation agrees with the notation used in Chapter 5.
We compute the first approximation of the Poincaré map. If $P \in S$, $H(P) = t$ is an initial point of the positive trajectory Γ of the perturbed system, then the first intersection of Γ with S is the value of the *return map*, $Q = \Phi(P)$ (see Figure 9). We calculate the increment $\Delta H = H(Q) - H(P)$ of the Hamilton function along Γ. Note that Γ is periodic iff $\Delta H = 0$ and it is hyperbolic stable (respectively unstable) limit cycle iff additionally $(\Delta H)'(P) < 0$ (respectively > 0). Using the representations $H'_x = -\dot{y} + \epsilon Q$, $H'_y = \dot{x} - \epsilon P$ we get

$$\begin{aligned} \Delta H &= \textstyle\int_0^T \dot{H} dt = \int (H'_x \dot{x} + H'_y \dot{y}) dt = \epsilon \int (H'_x P + H'_y Q) dt = \epsilon \int (Q\dot{x} - P\dot{y}) dt \\ &= \epsilon \textstyle\int_\Gamma Q dx - P dy. \end{aligned}$$

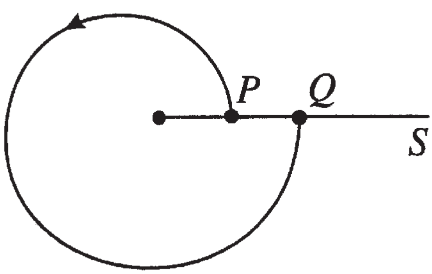

Figure 9

Because the phase curve Γ is close to the oval of $H = t$ (up to the order $O(\epsilon)$) we get

$$\Delta H = \epsilon I(t) + O(\epsilon^2)$$

where

$$I(t) = \int_{H=t} Qdx - Pdy$$

is the **Abelian integral**. In fact the path of integration is some real oval $\gamma(t)$ of the curve $H = t$ and the function I is defined in an interval $(t_{\min}, t_{\max})$ of t's, for which the ovals $\gamma(t)$ are compact and smooth.
(Probably this integral first appeared in the work [**Pon**] of L. S. Pontryagin. It was used intensively by V. K. Melnikov [**Mel**] as a tool for detecting sub-harmonic solutions in some periodic non-autonomous Hamiltonian systems. Some people (e.g. Arnold) claim that it was known already to Poincaré. Therefore in the literature it appears under different names: *Pontryagin integral, Poincaré–Pontryagin integral, Melnikov integral, Pontryagin–Melnikov integral, generating function.*)
We see that:

The necessary condition for existence of limit cycle $\gamma_\epsilon \to \delta(t_i)$ *is the equality* $I(t_i) = 0$.

Under some generic assumptions it is also a sufficient condition.

6.19. The weakened XVI-th Hilbert problem. *Consider the space of integrals* $I(t)$ *with* P, Q, H *polynomials of degree* $\leq n$ *and defined in the intervals* $(t_{\min}, t_{\max})$. *Find an estimate* $C(n)$ *for the number of zeroes of* $I(t)$ *uniform with respect to the polynomials* P, Q, H.

This problem (stated by V. I. Arnold [**Arn5**]) is also not solved completely, but there are many nice results concerning it.

6.20. Results. Firstly, A. N. Varchenko [**Var3**] and A. G. Khovanski [**Kh2**] proved that

$$C(n) < \infty,$$

i.e. existence of a uniform estimate. However they do not give any formula for $C(n)$. The proof of Varchenko is based on the methods developed in the book [**AVG**], (asymptotic expansions of integrals along cycles in complex algebraic curves), and some finiteness results from real analytic geometry. Khovanski observed that Abelian integrals belong to his class of Pfaff functions and applied his theory of fewnomials. Below we present some of the Varchenko–Khovanski arguments.
Concrete estimates are given with some restrictions on the Hamilton function.
In the case of the elliptic Hamiltonian $y^2 + x^3 - x$, G. S. Petrov [**Pet2**] proved the Chebyshev property of Abelian integrals. We present his beautiful proof below.
In the case of a hyperelliptic Hamiltonian $y^2 + R(x)$ (with fixed polynomial R) Petrov [**Pet3**] proved the linear estimate $\leq a \cdot n + b$ for the number of zeroes of any form $Qdx - Pdy$ of degree n.
For cubic H and quadratic P and Q, L. Gavrilov [**Gav1**] proved that the number of zeroes is ≤ 2 (also around two foci).
Other general estimates were obtained by Yu. Il'yashenko and S. Yu. Yakovenko ($\leq 2^{2^{cn}}$, $c = c(H)$ in [**IlY1**]) for generic Hamiltonians, by D. Novikov and

Yakovenko ($\leq 2^{cn}$ in [**NY**]) and by A. Glyutsuk and Yu. Il'yashenko ($\leq e^{2500n^2}$ for Hamiltonians with critical points of absolute value ≤ 1 in [**GY**]).
In 1996 A. G. Khovanski and G. S. Petrov announced the estimate

$$\leq a \cdot n + b,$$

where $a = a(\deg H)$ and $b = b(\deg H)$ depend only on the degree of the Hamiltonian (without an explicit formula). No restriction on H is made. This result is not yet published, but we present this proof below.

Below we present some estimates, with proofs, for the number of zeroes of $I(t) = I_\omega(t) \int_{H=t} \omega$ in the case of a polynomial 1-form

$$\omega = A(x, y)dx + B(x, y)dy$$

of degree $\deg \omega = \max(\deg A, \deg B)$ and with concrete H's. We begin with the quadratic Morse Hamiltonian.

6.21. Proposition. *If* $H = x^2+y^2$, *then* $I_\omega(t)$ *is a polynomial of degree* $\leq (n+1)/2$. *It has at most* $[(n+1)/2]-1$ *positive zeroes corresponding to eventual limit cycles.*

Proof. Consider the case when ω is homogeneous of degree j. Then, putting $x = \sqrt{h}\cos\theta$, $y = \sqrt{h}\sin\theta$, we get the trigonometric integral

$$I = t^{(j+1)/2} \int_0^{2\pi} R(\cos\theta, \sin\theta)d\theta,$$

where R is a homogeneous polynomial of degree $j+1$. This integral vanishes for odd $j+1$. □

Next is the case of the elliptic Hamiltonian

$$H = y^2 + x^3 - x$$

studied before. We integrate the *real* 1-form ω along the *real* oval $\gamma(t)$. The latter represents one of the two generators of the first homology group of the complex elliptic curve $\{H = t\} \subset \mathbb{C}P^2$. It vanishes at the critical point $x = 1/\sqrt{3}, y = 0$ with the critical value $t = -2/3\sqrt{3}$. The other generator is $\delta(t)$ and vanishes at $(-1/\sqrt{3}, 0)$ with the critical value $t = 2/3\sqrt{3}$ (see the points 5.19–5.24 and Figure 10).
The functions $I_\omega(t)$ have analytic prolongation to the complex arguments t. They are multivalued functions with unique branching points at $t = 2/3\sqrt{3}$. Thus, in the complex plane cut along the half-line $\{t \geq 2/3\sqrt{3}\}$, the functions I_ω are analytic and univalent. We denote by Ω the set

$$\mathbb{C} \setminus \Big(\{-2/3\sqrt{3}\} \cup [2/3\sqrt{3}, \infty)\Big)$$

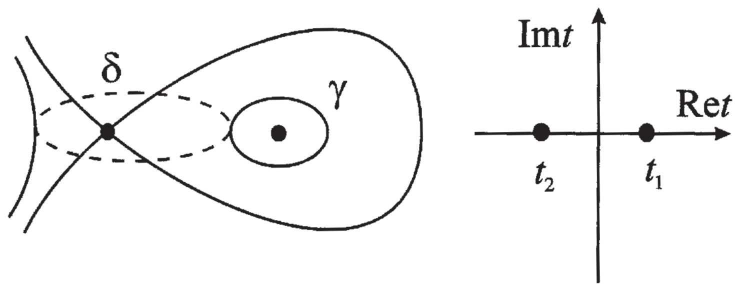

Figure 10

and consider I_ω as functions on Ω. We further note that Ω contains the interval $(-2/3\sqrt{3}, 2/3\sqrt{3})$ which is of interest for us. All these functions with $\deg \omega \le n$ form a finite dimensional vector space W_n.

6.22. Definition. A linear space W of functions defined on a set A, ($A \subset \mathbb{R}$, $\mathbb{C}$), is called **Chebyshev** iff any nonzero function from W has at most $\dim W - 1$ zeroes in A.
W is **Chebyshev with accuracy** k iff this number of zeroes is $\le \dim W - 1 + k$.

The reader can prove that in a k-dimensional linear space W of functions on A one can always choose a nonzero function vanishing at any $k-1$ previously chosen points.

6.23. Theorem of Petrov. (**[Pet2]**) *The space* $W_n = \{\int_{\gamma(t)} \omega : \ \omega$ real of degree $\le n$, $t \in \Omega\}$ *is Chebyshev.*

Proof. In 5.19–5.24 the following properties of the elliptic integrals were proved.

1. $I_\omega = P_0(t)I_0 + P_1(t)I_1$, *where* $I_j = \int_{\gamma(t)} x^i y dx$ *and* $P_{0,1}$ *are polynomials of degrees* $\le [(n-1)/2]$ *and* $\le [n/2] - 1$ *respectively. The space* W_n *has dimension equal to* n.
2. $5I_0 = 6tI_0' + 4I_1'$, $21I_1 = 4I_0' + 18hI_1'$.
3. $4(27t^2 - 4)I_1'' = 21I_1$.
4. $I_0 \sim t^{5/6}$, $I_1 \sim t^{7/6}$ *as* $t \to \infty$.

Below we prove additional properties of the elliptic integrals.

5. **Lemma.** *We have* $\operatorname{Im} I_1(t) \neq 0$ *for* $t > 2/3\sqrt{3}$.

Proof. In order to understand properly the statement of Lemma 5, we must recall the definition of I_1 as the analytic prolongation of an integral of a holomorphic form along the cycle $\gamma(t)$. At the point $t = 2/3\sqrt{3}$ this function has ramification but

the limit values of I_1 along the upper and the lower ridges of the cut $[2/3\sqrt{3}, \infty)$ are well defined. Thus the $\operatorname{Im} I_1$ is the imaginary value at the upper ridge.
Because the initial function (i.e. for $|t| < 2/3\sqrt{3}$) was real, I_1 behaves well under conjugation of the argument. In particular, the value of $\operatorname{Im} I_1$ at the lower ridge is equal to minus its value at the upper ridge. The function $\operatorname{Re} I_1$ is the same at both ridges.
The difference between values of I_1 at the upper ridge and lower ridge, i.e. $2 \operatorname{Im} I_1$, is equal to the variation of the integral as the value t varies around the critical value $2/3\sqrt{3}$. By the Picard–Lefschetz formula this variation is equal to the value of the form $xydx$ at the other generator of the first homology group $\delta(t)$. These arguments show that

$$\operatorname{Im} I_\omega(t) = \frac{1}{2} \int_{\delta(t)} \omega, \ t \geq 2/3\sqrt{3}$$

for any real form ω.
Therefore $z(t) = \operatorname{Im} I_1$ is expressed by means of integrals along cycles. In particular, it satisfies the equation (see 3.), i.e. $4(27t^2 - 4)z'' = 21z$. Moreover, $z(2/3\sqrt{3}) = 0$ (because δ vanishes there).
Because the factor $27t^2 - 4 > 0$ we have $z'' > 0$ iff $z > 0$ and $z'' < 0$ iff $z < 0$. Thus z is either positive and convex or negative and concave. In any case it cannot have zeroes. $\square$

6. **Lemma.** $I_1(t)$, $t < 2/3\sqrt{3}$, *vanishes only at the point* $t = -2/3\sqrt{3}$ *and this is a simple zero.*

Proof. Of course, $I_1(t)$ is real in this half-line and vanishes at $-2/3\sqrt{3}$. Moreover, because $I_1 = \int\int_{H \leq t} xdxdy$ and the domain $H \leq t$ is an approximate ellipse around the point $(1, 0)$ with the semi-axes $\sim \sqrt{t + 2/3\sqrt{3}}$, then $I_1 \sim (t + 2/3\sqrt{3})$ as $t \to -2/3\sqrt{3}$. This gives the simplicity of the zero.
The negativity of I_1 at the half-line $t < -2/3\sqrt{3}$ is proved in the same way as in the proof of Lemma 5.
The positivity of I_1 in the interval $(-2/3\sqrt{3}, 2/3\sqrt{3})$ needs application of some geometrical arguments. Note that the integral I_1 is proportional to the center of mass of the domain $H \leq t$. It is seen from Figure 10 that this center of mass lies in the right half of the plane. Also it is not difficult to show it analytically. $\square$

7. **Lemma.** $I_1(t)$ *has only one zero in the complex plane cut along* $[2/3\sqrt{3}, \infty)$.

Proof. Because I_1 is real on $\mathbb{R}$ and has only one zero on the real part of the domain (Lemma 6) the number of its zeroes is odd. It is enough to show that this number is < 3.
We use the *argument principle* which says that:

If a contour Γ *bounds some region of analyticity of a function* g, $g|_\Gamma \neq 0$, *then the number of zeroes of* g *in this domain is equal to the increment of the argument of* g *along* Γ *(divided by* 2π*).*

We take the contour as in Figure 11: a small loop around $t = 2/3\sqrt{3}$, a large loop around infinity and along the ridges of the cut.
The increment of $\arg I_1$ along the large circle is defined by the asymptotic of the integral at infinity (see 4.) and equals to $7/6$. Because $\operatorname{Im} I_1$ does not vanish along the cut, then the increment of $\arg I_1$ along the ridges and the small circle is ≤ 1. Thus $\Delta_\Gamma(\arg I_1) \leq 13/6 < 3$. $\square$

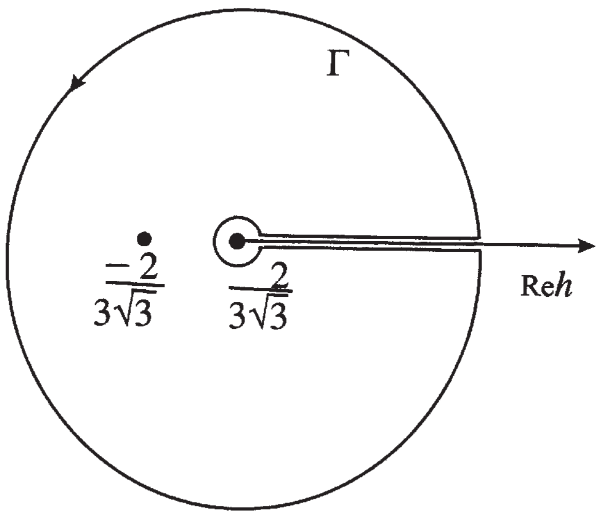

Figure 11

8. **Lemma.** *The function* $(I_0/I_1)(t)$, $t > 2/3\sqrt{3}$, *does not take real values.*

Proof. If that happened for $t = t_0$, then the vectors $\binom{\operatorname{Im} I_0}{\operatorname{Im} I_1}$ and $\binom{\operatorname{Re} I_0}{\operatorname{Re} I_1}$ would be parallel. However, both satisfy the same system of linear differential equations 2.; they form two solutions of it. The Wronskian of these solutions would vanish at t_0 and then it should vanish on the whole interval. This would mean that $(I_0/I_1)(t) \in \mathbb{R}$ for all $t > 2/3\sqrt{3}$. By the Schwarz reflection principle, I_0/I_1 could be prolonged to an analytic function in the plane outside $\{2/3\sqrt{3}\}$. Because it is bounded near $2/3\sqrt{3}$ this singularity would be removable and I_0/I_1 would be an integer function.
However, its exact asymptotic $\sim t^{-1/3}$ at infinity (see 4.) contradicts the above. $\square$

9. *Finishing of the proof of Theorem* 6.23.

Because $I_\omega(-2/3\sqrt{3}) = 0$ and I_1 has the only simple zero just at this point, then the number of zeroes of I_ω in the domain Ω is the same as the number of zeroes of the holomorphic function

$$I_\omega/I_1 = P_0(t) \cdot (I_0/I_1) + P_1(t)$$

in the plane cut along $[2/3\sqrt{3}, \infty)$.
Again we use the argument principle with the same contour Γ.
The increment of the argument along the large circle is $\sim \max(\deg P_0 - 1/3, \deg P_1)$. Along the ridges of the cut, the number of turns of I_ω/I_1 around 0 is bounded by the number of zeroes of the imaginary part of the function at the cut plus 1; (here we use the reality of the form ω). This gives $\leq \deg P_0 + 1$.
Summing it up (using 1.) we get the increment $\leq (n-1)$. □

6.24. Application to bifurcations. In the case of the Bogdanov–Takens bifurcation we have the elliptic integral

$$\int (\alpha x + \beta) y dx = \alpha I_0 + \beta I_1.$$

Thus $\deg P_{0,1} = 0$, the space of integrals is 2–dimensional and this elliptic integral has at most one zero corresponding to the unique limit cycle.

Below we present calculations of the number of zeroes of elliptic integrals appearing in bifurcations of periodic orbits with $1:2$ resonance ($q = 2$ in Example 6.15). In this case we show the divergence from the Chebyshev property (the accuracy grows with the degree). We also present applications of some real methods. We follow the work [**RZ**].
The Hamiltonian is also elliptic (see Figure 12)

$$H = y^2 \pm x^4 \pm 2x^2.$$

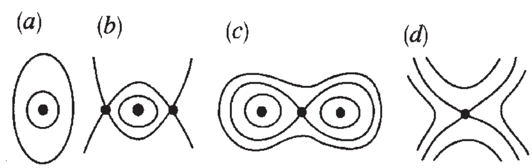

Figure 12

In the cases (a) and (b) the corresponding spaces of Abelian integrals are Chebyshev (Petrov).
We shall consider the case (c) with the symmetric form $\omega = Adx + Bdy$, i.e. the polynomials A, B contain only monomials of odd degree.
The Hamilton function $H = y^2 + x^4 - 2x^2$ has three critical points: $(\pm 1, 0)$ with the critical value $t = -1$ and $(0,0)$ with the critical value 0. At $(\pm 1, 0)$ two cycles γ, γ' vanish. We study the integral

$$\begin{array}{rcll} I_\omega(t) & = & \int_\gamma \omega + \int_{\gamma'} \omega = 2\int_\gamma \omega, & -1 \leq h < 0, \\ I_\omega(t) & = & \int_{\gamma''} \omega, & h > 0, \end{array}$$

where $\gamma'' = \gamma + \gamma'$ is the cycle presented in Figure 12(c).
Let $W_n = \{I_\omega : \deg \omega \le n, \omega \text{ is symmetric}\}$.

6.25. Theorem. ([**RZ**]) *We have* $\dim W_n = 2[(n-1)/2]+1$ *and the maximal number of zeroes of a nonzero function from* W_n *in the interval* $(-1,\infty)$ *is* $3[(n-1)/2]$.

Remark. We see that the accuracy of the Chebyshev property of W_n is about $n/2$ and grows. However, when one restricts the integrals to the interval $(0,\infty)$, then the Chebyshev property with the accuracy 1 holds (see [**RZ**]).

Proof of Theorem 6.25. 1. Define the integrals for $t \ge -1$,

$$I_0(t) = \int_{H=t} y dx, \quad I_1(t) = \int_{H=t} x^2 y dx.$$

We have the following properties (proved like the analogous properties of Petrov's elliptic integrals).

2. $I_\omega = P_0(t)I_0 + P_1(t)I_1$, *where* $P_{0,1}$ *are polynomials of degrees* $\le [(n-1)/2]$ *and* $\le [(n-1)/2]-1$ *respectively, thus* $\dim W_n = 2[(n-1)/2]+1$.

3. $3I_0 = 4tI_0' + 4I_1'$, $15I_1 = 4tI_0' + (12t+16)I_1'$ (Picard–Fuchs equations).

4. $I_0(t) = c_1 + c_2 t \ln|t|^{-1} + \ldots$, $I_1(t) = d_1 + d_2 t \ln |t|^{-1} + \ldots$ *as* $t \to 0$, *with* $c_{1,2}, d_{1,2} > 0$.

Denote
$$Q = I_1/I_0.$$

The properties 3 and 4 imply the following.

5. $Q(t) = e_1 + e_2 t \ln|t|^{-1} + \ldots$ *as* $t \to 0$.

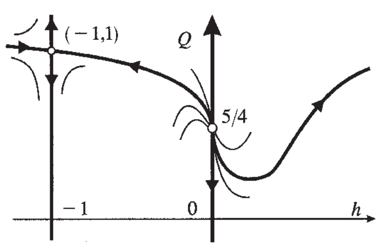

Figure 13

6. $4h(t+1)Q' = 5Q^2 + 2tQ - 4Q - t$ *and the graph of the function* $t \to Q(t)$ *consists of the phase curves of the vector field*

$$\dot{t} = 4t(t+1), \quad \dot{Q} = 5Q^2 + 2tQ - 4Q - t \tag{2.1}$$

indicated in Figure 13.

The linear part of the vector field (2.1) at the singular point $(0, 4/5)$ is the Jordan cell $\begin{pmatrix} 4 & 0 \\ 3/5 & 4 \end{pmatrix}$ and all trajectories near it have the form $Q = 4/5 + ct\ln|t| + \dots$. The singular point $(-1, 1)$ is a saddle with the linear part $\begin{pmatrix} -4 & 0 \\ 1 & 6 \end{pmatrix}$ and our graphic lies in two separatrices of this saddle.

7. $I_0 \sim t^{3/4}$, $I_1 \sim t^{5/4}$, $Q \sim t^{1/2}$ *as* $t \to \infty$.

8. $Q(t)$ *is decreasing for* $t < 0$.

Proof. We have $Q'(-1) < 0$, $Q'(0) = -\infty$, $Q''(0) = -\infty$ and $Q'(t) < 0, Q''(t) < 0$ as $t \to -\infty$.
On the other hand, $(4t(t+1)Q')'|_{Q'=0} = 2Q - 1$. But $\dot{Q}|_{Q=1/2} = -3/4 < 0$, which implies $Q(t) > 1/2$. This means that the function $Q(t)$ would be concave at each critical point in the interval $(-1, 0)$ and convex at critical points in $(-\infty, -1)$. This, combined with the behaviour at the endpoints, gives the result. □

9. $Q(t)$ *has a unique minimum at* $t_* > 0$.

Proof. From the asymptotic behaviors at $t = 0$ and at $t = \infty$ it follows that such minimum exists. The condition $Q' = Q'' = 0$ means intersection of the line $2Q = 1$ and the hyperbola $2Q^2 + 2tQ - 4Q - t = 0$. We have shown that this intersection is empty, which means that Q'' has the same sign at each critical point of Q. □

Let

$$g(t) = \frac{I_1}{I_0} + \frac{P_0}{P_1} = Q + R.$$

Its zeroes are the zeroes of I_ω which are > -1.

10. *We have* $4P_1^2 t(t+1)g'|_{g=0} = S(t)$ *where* $S(t)$ *is a polynomial of degree* $\leq 2[(n-1)/2]$.

11. *Upper bound for the number of zeroes.* Here we use the idea of Petrov from **[Pet1]**.
We divide the interval $(-1, \infty)$ into the subintervals of continuity of g. The number of such subintervals is $\leq \deg P_1 + 1$. In each such subinterval we apply the *Rolle principle* to the function g:

Between any two zeroes of a function a zero of the derivative lies.

Thus, by 10, between two zeroes of g there is either a zero of $S(t)$ or the point $t = 0$. On the other hand, the zeroes of $g'|_{g=0}$ are the points of contact of the vector field (2.1) with the curve $Q = -R(t)$. There is such a contact point between the line $t = -1$ and the first zero of the function g.
We have then

$$\#\{g = 0\} \leq [(\#\{S = 0\} - 1) + 1] + \#(\text{intervals}) \leq 3[(n-1)/2].$$

12. *Lower bound.* The function I_ω has the following expansion in a neighborhood of the point $t=0$,

$$I_\omega(t) = b_0 + a_1 t\ln|t|^{-1} + b_1 t + a_2 t^2 \ln|t|^{-1} + b_2 t^2 + \dots .$$

If $t>0$, then the t^i and $t^i\ln|t|^{-1}$ are positive. We have also

$$I_\omega(-t) = b_0 - a_1 t\ln|t|^{-1} - b_1 t + a_2 t^2 \ln|t|^{-1} + b_2 t^2 - \dots .$$

We use the *Descartes principle*:

The number of positive zeroes of a polynomial is bounded by the number of sign changes of its coefficients. Moreover, this bound is achieved for a suitable choice of the absolute values of the coefficients.

In fact, this principle holds also in the case when the polynomial is replaced by a finite sum of functions like t^i and $t^i\ln|t|^{-1}$ and their small perturbations.
We apply this principle to the cases of functions $I_\omega(t)$ and $I_\omega(-t)$ where the coefficients are chosen in a way to get the maximal number of zeroes. Thus we have the leading coefficient $a_m = O(1)\neq 0$ (or $b_m = O(1)\neq 0$) and the absolute values are chosen in a way to give the number of zeroes in the intervals $t<0$ and $t>0$ prescribed by the sign changes, i.e.

$$0<|b_0| << |a_1| << |b_1| << |a_2| << \dots << |a_m|.$$

Note the following property:

The sign change between a_i and b_i in $I(t)$ leads to a corresponding sign change in $I(-t)$ and the sign change between b_i and a_{i+1} implies no sign change in $I(-t)$.

Let k_1 be the number of sign changes in $I(t)$ implying the sign changes in $I(-t)$ and k_2 be the number of remaining sign changes. The number of positive zeroes of I_ω is k_1+k_2
If $a_m\neq 0$ is the leading coefficient, then the number of sign changes in $I(-t)$ is $k_1+(m-k_2)$. The total number of small zeroes of I is $2k_1+m$, where $k_1\le m-1$. This gives the maximal number $3m-2$ of zeroes.
If $b_m\neq 0$ is the leading coefficient, then the same arguments give the maximal number $3m$ of zeroes.
It remains to show that the coefficients a_i, b_i in the expansion of the integral are controlled by the coefficients of the polynomials $P_{0,1}$ in the expansion 2. Indeed, using the properties 4, we see that for $P_0 = at^m$, $m=[(n-1)/2]$, $P_1\equiv 0$ the first nonzero coefficient in the asymptotic expansion of I is b_m. Taking a suitable $P_0 = at^{m-1}$ or $P_1 = bt^{m-1}$ we get a_m as leading. Other coefficients are controlled in the same way.
Theorem 6.25 is complete. □

Below we present the promised proof of the linear bound which is not published yet. The initial claim of Petrov and Khovanski was the estimate $an+b$, where the constant a can be chosen universal, not depending on n and H, and b depends on H. The below proof is based on the conception of rational envelopes of Abelian integrals developed in [**Yak**] and [**IlY1**], on the lecture of Petrov in the Banach Center in Warsaw (1995) and on estimates in the style of Khovanski [**Kh2**] and Varchenko [**Var3**].

6.26. Theorem of Petrov and Khovanski. *The number of zeroes of any Abelian integral I_ω along a family of real ovals of the curve $H = t$ with $\deg \omega = n$ is bounded by*

$$a \cdot n,$$

where $a = a(\deg H)$ is a constant depending only on $\deg H$.

Proof. 1. We begin with a representation of I_ω as a combination of given Abelian integrals with rational coefficients. This proposition (in a restricted form) was proved by S. Yu. Yakovenko in [**Yak**] . His proof uses the theorem of Röhrl and Plemelj (Theorem 8.37 in Chapter 8) about realization of a given monodromy group by means of a linear rational differential system. Our proof is different and without restrictions, i.e. the critical points of H can be non-isolated and may lie at infinity.

2. **Lemma (Rational envelopes).** *There exist real polynomial holomorphic 1-forms $\omega_1, \ldots, \omega_k$ such that for any real polynomial holomorphic 1-form ω of degree n the following representation holds:*

$$P_0(t)^K \cdot I_\omega = P_1(t) I_{\omega_1} + \ldots + P_k(t) I_{\omega_k}.$$

Here the integer $k = k(d)$ depends only on $d = \deg H$, P_i are real polynomials such that P_0 depends only on H, $\deg P_i \leq c \cdot n$, $i > 0$, the positive integer $K \leq c \cdot n$ and the constant c depends only on $d = \deg H$.

Proof. (a) Fix a Hamilton function H. We take the family of complex level curves $\{H = t\} \subset \mathbb{C}^2$. They form the locally trivial fibration (an analogue of the Milnor fibration)

$$\mathbb{C}^2 \setminus H^{-1}(\textit{atypical values}) \xrightarrow{H} \mathbb{C} \setminus (\textit{atypical values}).$$

Here by the *atypical values* we mean the usual values of H at the critical points and the values corresponding to the 'bad' behaviour of the family of levels at infinity.

The generic fibre of this fibration is an open Riemann surface, whose topological type is a bucket of circles. Its first homology group is generated by concrete geometric cycles $\delta_1(t), \ldots, \delta_k(t)$.

Fix an atypical value t_0 and the cycles $\delta_i(t_0)$. We choose real polynomial forms $\omega_1, \ldots, \omega_k$ such that their restrictions to the curve $H = t_0$ generate $H^1(\{H = t_0\})$. It is done by a suitable approximation of holomorphic generators of the first de Rham cohomology group in such a way that $\det \left(\int_{\delta_i(t_0)} \omega_j \right) \neq 0$. Later we will impose other restrictions for the forms ω_i.

(Such polynomial generators of the de Rham cohomology of an affine algebraic variety exist in the general case (A. Grothendieck). This follows from so-called quasi-equivalence between the holomorphic de Rham complex $\Omega^\bullet$ and the complex $\Omega^\bullet_{alg}$ of holomorphic forms with regular behaviour at infinity (see the point 7.33(e) in Chapter 7 and [**GH**]).)

Repeating the proof of Corollary 5.28(b), we obtain the following, locally unique, representation

$$\int_{\delta(t)} \omega = \sum_j p_j(t) \int_{\delta(t)} \omega_j \tag{2.2}$$

for any polynomial holomorphic form ω and any family $\delta(t)$ of integer cycles in $H = t$. The functions $p_j(t)$ do not depend on δ and are prolonged to holomorphic and single-valued functions in

$$\mathbb{C} \setminus \{t_1, \dots, t_r\},$$

where t_j are atypical values. Each function $p_j(t)$ is expressed as a ratio of two determinants, with the denominator equal to $\det\left(\int_{\delta_i} \omega_j\right)$ (the same for all p_j) and with the numerator equal to the determinant of the matrix $\int_{\delta_i} \tilde{\omega}_j$ obtained from the previous one by replacing the j-th column by $\int_{\delta_i} \omega$; $\tilde{\omega}_j = \omega$, $\tilde{\omega}_l = \omega_l$, $l \neq j$.

The atypical values t_i are of four types:

(i) the critical values of isolated critical points of H in the finite plane $\mathbb{C}^2$;

(ii) the critical values of non-isolated critical points of H in $\mathbb{C}^2$;

(iii) the critical values of critical points at infinity (in $\mathbb{C}P^2$);

(iv) the zeroes of the determinant $\det \int_{\delta_i} \omega_j$.

In Chapter 5 it was shown that the Abelian integrals are regular near the critical values of the type (i). Thus $p_j(t)$ are meromorphic near them, $p_j = q_j(t)/(t-t_j)^{K_j}$, where the power K_j depends on H and on the forms ω_i and q_j is holomorphic. The same statement is true near the points of the type (iv).

For the points of the type (ii) we apply the resolution of the (non-isolated) singularities in the finite plane. In some local analytic coordinates $\tilde{x}, \tilde{y}$ we obtain $H - t_j = \tilde{x}^p \tilde{y}^q$. If $(p,q) = gcd(p,q) = 1$ then the Riemann surface $\tilde{x}^p \tilde{y}^q = t - t_j$ contains the vanishing cycle $\tau(t) : \quad \tilde{x}(\theta) = \epsilon e^{iq\theta}$, $\tilde{y} = \epsilon e^{-ip\theta}$, $0 \leq \theta \leq 2\pi$, $\epsilon = (t - t_j)^{1/(p+q)}$. It is easy to estimate the integrals along $\tau(t)$. If p and q are not relatively prime then several such cycles vanish.

Near a smooth part E of the set of non-isolated critical points, i.e. where $H - t_j = \tilde{x}^p$, we do not have vanishing cycles. Here p local components of $H = t$ approach E and some part of a cycle δ_i may pass near E. The corresponding contributions to the integrals are easy to estimate.

As an example of the function $x(1 - x(y^2 + 1))$ shows, the critical points of H may lie on the line $L_\infty = \{(x : y : 0)\} \subset \mathbb{C}P^2$ (at infinity). We include also the value $t = \infty$ into the part (iii) of the set of critical values. The foliation of $\mathbb{C}^2$ into the level curves $H(x, y) = t$ is a holomorphic foliation which can be prolonged to a holomorphic foliation $\mathcal{F}$ in $\mathbb{C}P^2$ (see Chapter 9 below). Near a critical point $(x : y : z) = (x_0 : y_0 : 0)$ at infinity, the function H is rational, equal to $F_m(u, z)/z^m$, $m = \deg H$ (u and z are local projective coordinates). If the curves $H = t$ have only isolated (transversal) intersections with L_∞, then these points are singular points of the foliation $\mathcal{F}$ of the node type (see Chapter 9 below).

For the points with bad behaviour of the $\mathcal{F}$ we apply the resolution process: by Theorem 9.18 in Chapter 9, such resolution exists. We obtain either $H - t_j = \tilde{x}^p \tilde{y}^q$ (a saddle point) or $H - t_j = \tilde{x}^p z^{-m}$ (a node). In the first case a certain cycle vanishes. In the second case the local Riemann surfaces $H = t$ are punctured discs, with one cycle $\sigma(t)$ generating its local first homology group. However, the cycle $\sigma(t)$ does not intersect the cycles which have nontrivial monodromy and are in some way associated with the real ovals of $H = t$. σ is monodromy invariant and the integral $\int_\sigma \eta$ equals the residuum of the form η at the point $\tilde{x} = z = 0$.

The above analysis shows that we have to study the asymptotics of the integrals along those cycles $\delta(t)$, which become large as $t \to t_j$ or as $t \to \infty$. Here we can either use the general result about regularity of the Gauss–Manin connection (see Remark 5.16 above or Remark 8.19 below) or apply a priori estimates (as in [**Yak**]). We recall Yakovenko's arguments.

The cycles $\delta_i(t)$ can be chosen as lifts to the Riemann surface of the algebraic function $y(x)$ (defined by $H(x, y) = t$) of some loops $\gamma_i(t)$ in the x-plane, deprived of the ramification points x_k of the function $y(x)$. The points $x_k = x_k(t)$ vary regularly, with power type escape to infinity. This implies that the absolute values of the coordinates along the cycles $\delta_i(t)$ also have regular growth.

Because the forms ω and ω_i are polynomial their integrals along the cycles $\delta_i(t)$ also have polynomial growth. The coefficient functions $p_i(t)$ (from the expansion $\int \omega = \sum p_i \int \omega_i$) are expressed as ratios of determinants, which are regular. This implies that $p_i(t) = P_i(t)/P_0(t)^K$ where P_i are polynomials and $P_0 = \prod(t - t_i)$. It is also clear that the degrees of $P_1, P_2, \ldots$ grow linearly with $n = \deg \omega$ and the exponent K is linear in n.

(b) So the proposition is proved, but with constants depending on the Hamilton function H. We need some arguments which would show uniformity of these constants.

The space of real polynomial Hamiltonians of degree $\leq d$ (and without the constant term) can be identified with a sphere $\mathcal{H} \subset \mathbb{R}^N$, $\mathcal{H} = S^{N-1}$, $N = (d+1)(d+2)/2-1$: the Hamilton functions H and λH lead to the same Abelian integrals. So, it is enough to show that the estimates are uniform with respect to H, locally in $\mathcal{H}$.

$\mathcal{H}$ is a real algebraic variety; its complex variant $\mathcal{H}^{\mathbb{C}}$ is a quasi-projective variety (a quadric). Consider the space $\mathcal{H}^{\mathbb{C}} \times \mathbb{C}$; (its points (H, t) correspond to algebraic curves $H(x, y) = t$). Let $\mathcal{M}^{\mathbb{C}} = \overline{\mathcal{H}^{\mathbb{C}}} \times \mathbb{C}P^1$ be the closure of this set; (here $\overline{\mathcal{H}^{\mathbb{C}}}$ is the projective closure). Denote $\Sigma_1^{\mathbb{C}} = \mathcal{M}^{\mathbb{C}} \setminus \mathcal{H}^{\mathbb{C}} \times \mathbb{C}$; it is a hypersurface in $\mathcal{M}^{\mathbb{C}}$.

(We agree to denote complex varieties by the upper index $\mathbb{C}$, their real parts are denoted without this index.)
The degenerate curves $H = t$ correspond to points from a bifurcational subset $\Sigma_2^{\mathbb{C}} \subset \mathcal{H}^{\mathbb{C}} \times \mathbb{C}$. $\Sigma_2^{\mathbb{C}}$ is a proper algebraic hypersurface. The restriction of the projection $\mathbb{C}^2 \times \left(\mathcal{H}^{\mathbb{C}} \times \mathbb{C}\right) \to \mathcal{H}^{\mathbb{C}} \times \mathbb{C}$ to $\{(x, y, H, t) : H(x, y) = t,\ (H, t) \notin \Sigma_2^{\mathbb{C}}\}$ is a locally trivial (Milnor) fibration over $\mathcal{M}^{\mathbb{C}} \diagdown \left(\Sigma_1^{\mathbb{C}} \cup \Sigma_2^{\mathbb{C}}\right)$.
The polynomial forms ω_i, $i = 1, \dots, k$ are defined as the same for all $H \in \mathcal{H}^{\mathbb{C}}$. For generic $(H, t) \in \mathcal{M}^{\mathbb{C}} \diagdown \left(\Sigma_1^{\mathbb{C}} \cup \Sigma_2^{\mathbb{C}}\right)$ the cohomology classes of the forms ω_i in $H^1(\{H = t\})$ form the basis of this space. The integrals $\int_{\delta_i(H,t)} \omega_j$ and $\int_{\delta_i(H,t)} \omega$ are well defined for such (H, t); because the cycles $\delta_i = \delta_i(H, t) \subset \{H = t\}$ are well defined there. Denote by $\Sigma_3^{\mathbb{C}} \subset \mathcal{M}^{\mathbb{C}} \diagdown \left(\Sigma_1^{\mathbb{C}} \cup \Sigma_2^{\mathbb{C}}\right)$ the set of those (H, t) for which $\det \int_{\delta_i(H,t)} \omega_j = 0$. It is an analytic subset; (as we shall see it is also an algebraic subset of $\mathcal{M}^{\mathbb{C}}$). Denote also $\Sigma^{\mathbb{C}} = \Sigma_1^{\mathbb{C}} \cup \Sigma_2^{\mathbb{C}} \cup \Sigma_3^{\mathbb{C}}$.
The integrals, treated as functions of (H, t), have singularities along $\Sigma_1^{\mathbb{C}} \cup \Sigma_2^{\mathbb{C}}$. These singularities are investigated using the resolution of singularities of this variety. Let $\pi : \widetilde{\mathcal{M}^{\mathbb{C}}} \to \mathcal{M}^{\mathbb{C}}$ be the resolution of singularities of $\Sigma_1^{\mathbb{C}} \cup \Sigma_2^{\mathbb{C}}$.
Near any point of $\pi^{-1}(\Sigma_1^{\mathbb{C}} \cup \Sigma_2^{\mathbb{C}})$ there exists a local system of coordinates $z_1, \dots, z_N$ such that $\Sigma_1^{\mathbb{C}} \cup \Sigma_2^{\mathbb{C}} = \{z_1 \cdot z_2 \cdot \ldots \cdot z_m = 0\}$. The integrals are treated as functions of $z = (z_1, \dots, z_N)$.
They admit the finite expansions

$$\sum_{k,\alpha} a_{k,\alpha}(z) \cdot z_1^{\alpha_1} \left(\ln z_1\right)^{k_1} \cdot \ldots \cdot z_m^{\alpha_m} \left(\ln z_m\right)^{k_m}, \tag{2.3}$$

where the (finite) sum runs over integer vectors $k = (k_1, \dots, k_m)$ and rational vectors $\alpha = (\alpha_1, \dots, \alpha_m)$. The exponents α_l are such that $e^{2\pi i \alpha_l}$ are eigenvalues of the monodromy operators M_l associated with the loops $\{z_l = \epsilon e^{i\theta},\ z_j = const\ (j \neq l)\}$ (around the hypersurfaces $\{z_l = 0\}$). The integers k_l take values 0 or 1. The coefficients $a_{k,\alpha}(z)$ are analytic functions. The proof of this expansion is the same as the proof of Theorem 5.14 (with use of the regularity of integrals).
The determinants $\det\left(\int_{\delta_i(t,H)} \omega_j\right)$ and $\det\left(\int_{\delta_i(t,H)} \tilde{\omega_j}\right)$ (where $\tilde{\omega_j} = \omega_j$ or $= \omega$) have simple monodromy properties. They remain fixed or change sign when the z turns around the divisor $z_l = 0$ (as in the proof of Theorem 5.25). They take the form $\sqrt{z_{l_1} \dots z_{l_r}} \cdot \phi(z)$, where ϕ is meromorphic. This shows that $\Sigma_3^{\mathbb{C}} = \left\{\det\left(\int_{\delta_i(t,H)} \omega_j\right) = 0\right\}$ has algebraic singularities, and hence is algebraic.
The formula (2.2) is generalized as

$$\int_{\delta(t,H)} \omega = \sum p_j(t, H) \int_{\delta(t,H)} \omega_j$$

for $(H, t) \in \mathcal{M}^{\mathbb{C}} \diagdown \Sigma^{\mathbb{C}}$. The functions $p_j(t, H)$ are holomorphic, single-valued and regular. Thus they are restrictions of rational functions on $\mathcal{M}^{\mathbb{C}}$, with poles at $\Sigma^{\mathbb{C}}$. Take one function p_j. We restrict it to the real lines $L_H = \{(H, t),\ t \in \mathbb{R}\}$, $p_j|_{L_H} = p_j(\cdot, H)$. There are three possibilities:

(i) $p_j(\cdot, H) \equiv 0$,

(ii) $p_j(\cdot, H) \equiv \infty$,

(iii) $p_j(\cdot, H) = P_j(t)/Q_j(t)$ is a nontrivial rational function.

We must eliminate the case (ii). It corresponds to the situation when $\det\left(\int_{\delta_i} \omega_j\right)|_{L_H} \equiv 0$. We achieve the goal by imposing additional restrictions on the forms ω_i.

We deal with two situations: (α) the line $L_H \not\subset \Sigma_2$ (the real part of $\Sigma_2^{\mathbb{C}}$) and (β) $L_H \subset \Sigma_2$. In the case (α) we deal with irregular position of the component Σ_3 with respect to the fibers of the projection $\mathcal{H} \times \mathbb{R} \to \mathcal{H} : L_H \subset \Sigma_3$. It is clear that the opposite property, $L_H \not\subset \Sigma_3$, is an open condition with respect to the forms $\omega_1, \dots, \omega_k$.

In the case (β) all the curves $\{H(x,y) = t,\ t \in \mathbb{R}\}$ are non-generic; (e.g. when $\deg H < d$). It is not surprising that there $\det \int_{\delta_i} \omega_j|_{L_H} \equiv 0$, but then also $\det \int_{\delta_i} \tilde{\omega_j}|_{L_H} \equiv 0$. Here $p_j(t,H)$ is understood as a limit of the ratio of determinants $\det \int_{\delta_i} \tilde{\omega}_j / \det \int_{\delta_i} \omega_j$ at points $(H', t') \in \mathcal{H} \times \mathbb{R} \setminus (\Sigma_2 \cup \Sigma_3)$, $(H', t') \to (H, t)$. This shows that we have to compare the asymptotic expansions of two determinants. The condition that the expansion of the $\det \int_{\delta_i} \omega_j$ is optimal (i.e. with smallest exponents) at a general point of the line $L_H \subset \Sigma_2$ is an open condition for $\omega_1, \dots, \omega_k$.

Using the openness of the above conditions for the 1-forms and the compactness of the set of lines L_H, $H \in \mathcal{H}$, we see that there exist forms $\omega_1, \dots, \omega_k$ such that $p_j|_{L_H} \not\equiv \infty$.

(c) Let us pass to the estimation of the constant c from the proposition (in $\deg P_i \leq cn$ and $Q = P_0^K$, $K \leq cn$). It is clear that it is enough to get a local uniform estimate for the asymptotic as $n = \deg \omega \to \infty$ of the integral $\int_{\delta(t,H)} \omega$ with respect to ω and H. Here the forms ω belong to the compact space $\Omega = \{\sum_{i+j \leq n} x^i y^j (a_{ij} dx + b_{ij} dy) : a_{ij}, b_{ij} \in \mathbb{R}, \sum a_{ij}^2 + b_{ij}^2 = 1\}$ and the Hamiltonians H belong to the compact space $\mathcal{H}$.

This uniform estimate follows from the expansion (2.3) (at points from $\Sigma_1 \cup \Sigma_2$), where the exponents α_j are of order $O(n)$ and the coefficients $a_{k,\alpha}(z)$ are uniformly bounded, locally with respect to z and globally with respect to $\omega \in \Omega$. The bound $|\alpha_j| < \mathrm{const} \cdot n$ is proved using the arguments from the point (a) above. We represent the cycles $\delta_i(t,H)$ as lifts (to the Riemann surface of the algebraic function $y(x)$ defined by $H(x,y) = t$) of loops $\gamma_i(t,H)$ in the x-plane deprived of the ramification points $x_j = x_j(t,H)$. The points $x_j = x_j(z)$ escape to infinity in a regular way as $z \to 0$. □

3. **The linear estimate when H has only real critical values.** It is enough to study the zeroes of the real polynomial envelope $P_1 I_1 + \dots P_k I_k$, or of

$$g(t) = P_1 + P_2 \frac{I_2}{I_1} + \dots + P_k \frac{I_k}{I_1}.$$

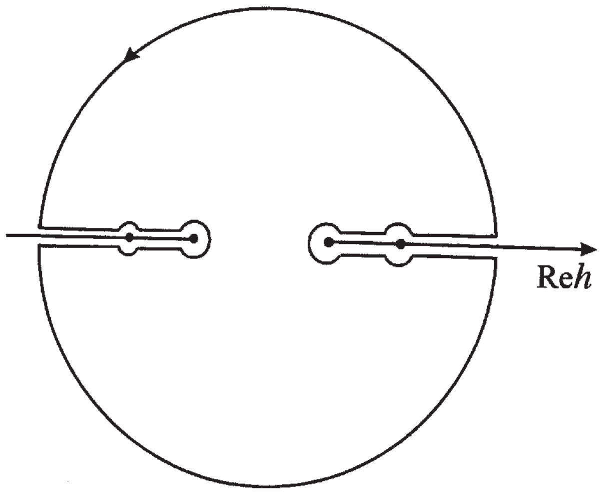

Figure 14

We apply the argument principle to the contour presented in Figure 14. The increment of the argument along the large circle is bounded by const$\cdot n$. Along the small semi-circles around the singular points t_i the increment of the argument is uniformly bounded; note that they are negatively oriented. The increment along the cuts is bounded by the number of zeroes of the imaginary part of g at the cuts. The latter imaginary parts are proportional to the combinations

$$P_2K_2 + \ldots + P_kK_k,$$

where K_j are expressed as some simple functions of integrals of ω_i along the cycles δ_j (i.e. we can put $K_j = \int_{var\,\delta} \omega_j \cdot \int_\delta \omega_1 - \int_\delta \omega_j \cdot \int_{var\,\delta} \omega_1$). This is done as in the proof of Theorem 6.25. Here we use the reality of I_j's.

Therefore, the problem is reduced to the estimation of the number of zeroes of a polynomial envelope of $k-1$ functions K_i which have regular singularities at the critical points of H and are real at an interval of the real axis.

We take the function $P_2 + P_3(K_3/K_2) + \ldots + P_k(K_k/K_2)$ and repeat the above estimation of its argument along contours as in Figure 14.

After a finite number of such steps we obtain the estimate $a(d) \cdot n + b(H)$ for the number of zeroes of the Abelian integral. Here the constant $b = b(H)$ is approximately equal to the number of zeroes (in $\mathbb{R} \setminus \{t_1, \ldots, t_r\}$) of an expression $A = A(t)$, which depends on the integrals $\int_{\delta_i} \omega_j$. A is a combination of products of such integrals.

Here we find that b depends on H; in the points 5, 6, 7 below we shall show that b depends only on $d = \deg H$.

4. **The linear estimate in the general case.** Assume that H has non-real critical values.
We construct a polynomial F such that the composition $F \circ H$ has only real critical values. The function F is a composition of functions of the form $t \to (t-a_j)^2$, where $a_j \pm ib_j$, $b_j \neq 0$ is a critical value of the function obtained in the previous step. Then the composed function acquires a real critical value $-b_j^2$. After a finite number of steps we eliminate all non-real critical values.
Thus $F = F_m \circ \ldots \circ F_1$, $F_i(t) = (t-a_i)^2$. The values of the composition $F \circ H$ are denoted by s.
If $m = 1$ then we have $t = a_1 + \sqrt{s}$. If $m = 2$ then $t = a_1 + \sqrt{a_2 + \sqrt{s}}$. Generally $t = a_1 + \sqrt{a_2 + \sqrt{a_3 + \ldots + \sqrt{s}}}$.
Denote $\Phi_m(s) = \sqrt{s}$, $\Phi_{m-1}(s) = \sqrt{a_{m-1} + \sqrt{s}}, \ldots, \Phi_1 = \sqrt{a_2 + \sqrt{a_3 \ldots + \sqrt{s}}}$ and $\tilde{I}_j(s) = I_j(t)$. The functions Φ_k and $\tilde{I}_j$ are multivalued holomorphic functions with singularities at the set of critical values of $F \circ H$. These singularities are regular (of power type).
Next, the polynomials P_j (from the proposition about rational envelopes) are expressed as polynomial combinations of products of the functions Φ_j.
This shows that the problem reduces itself to the problem of estimation of the number of zeroes of the combination

$$Q_1(s)J_1 + \ldots + Q_p(s)J_p,$$

where Q_j are polynomials of degrees $\leq \mathrm{const}\cdot n$ and J_i are multivalued holomorphic functions with regular real singularities.
For such combinations one can apply the proof from the previous case.
In this point we introduced some definitions and results which are used only in this proof. Therefore they will appear unnumbered.

5. **Towards an estimate of the constant** $b(H)$. Because any form of degree $n = 0$ is exact, its integral vanishes and $\#\{\text{isolated zeroes}\}|_{n=0} = 0$. This means that we should show that $b(H)$ is uniformly bounded with respect to $H \in \mathcal{H}$, where $\mathcal{H}$ is the space of Hamiltonians of degree $\leq d$.
Here we follow the proof of existence of a general estimate $\#\{I_\omega = 0\} \leq C(d,n)$ given by A. N. Varchenko in **[Var3]**. The ingredients of the proof are the following: the asymptotic expansion (2.3) (in its real form corresponding to the real resolution of a real algebraic variety), Khovanski's theory **[Kh2]** (of estimation of the number of solutions of a Pfaff system by the number of solutions of an analytic system) and the local uniform estimation of the number of solutions of analytic systems depending on parameters (theorem of Gabrielov **[Gab]**).
Recall that we have to estimate the number of zeroes of a function $A(s) = A(s;H)$, which is a combination of products of Abelian integrals (of forms ω_i) and of the functions Φ_j. Here the argument is s, where $s = (\ldots((t-a_1)^2 - a_2)^2 \ldots - a_m)^2 = F(t)$. This suggests that we have to replace the space $\{(H,t)\} = \mathcal{H} \times \mathbb{R}$ by the space

$\mathcal{N}$ of pairs (H,s). So, we have the algebraic mapping $\Psi : \mathcal{M} = \mathcal{H} \times \mathbb{R}P^1 \to \mathcal{N}$ and a new "bifurcational" subset $\Lambda \subset \mathcal{N}$. Λ is the union of $\Psi(\Sigma_1 \cup \Sigma_2)$ and of the set of critical values of Ψ. The mapping Ψ can be extended to a complex holomorphic mapping from a neighborhood of $\mathcal{M}$ in $\mathcal{M}^{\mathbb{C}}$ to a neighborhood of $\mathcal{N}$ in its complex analogue $\mathcal{N}^{\mathbb{C}}$.

We apply resolution of singularities of the real hypersurface Λ. Thus we have a real manifold $\widetilde{\mathcal{N}}$ with a resolution mapping $\widetilde{\mathcal{N}} \to \mathcal{N}$. The function A can be treated as a function on $\widetilde{\mathcal{N}}$. Near points from $\widetilde{\Lambda}$ (the inverse images of Λ) A admits asymptotic expansion of the type (2.3).

We cover the (compact) variety $\widetilde{\mathcal{N}}$ by finitely many charts U_i, defined by analytic inequalities, such that either (i) $U_i \cap \widetilde{\Lambda} = \emptyset$ or (ii) A admits expansion (2.3) in U_i. In charts of the type (i) the function $A(s;H)$ is real analytic.

In charts of the type (ii) we have (after eventual multiplication by a nonzero function)

$$A = A_0(z_1,\dots,z_N;(-\ln z_1)^{-1},\dots,(-\ln z_m)^{-1}; z_1^{\alpha_{11}},\dots,z_1^{\alpha_{1r_1}}; z_2^{\alpha_{21}},\dots,z_m^{\alpha_{mr_m}}).$$

Here the function $A_0(z_1,\dots,z_N;v_1,\dots,v_m;u_{11},\dots,u_{mr_m}) = A_0(z,v,u)$, $z_1,\dots,z_m > 0$ and small, $z_{m+1},\dots,z_N$ small, is analytic in all variables. Note that the quantities $v_i = (-\ln z_i)^{-1}$ and $u_{ij} = z_i^{\alpha_{ij}}$, $i \le m$, are also positive and small; because α_{ij} are rational and > 0 (we can assume it). We assume (for definiteness) that $z_i, v_i, u_{ij} \in (0,1)$.

We have the analytic map $\Pi : U_i \to \mathcal{H}$. Our task is to estimate the cardinality of $\{A=0\} \cap \Pi^{-1}(H)$ locally uniformly with respect to H, e.g. for $H \in V_i$ where $V_i = \{H \in \mathcal{H} : |H - H_0|^2 \le \theta\}$ is a ball in $\mathcal{H}$. For the charts of the type (i) we have a problem of estimation of local solutions of an analytic equation depending analytically on parameters. In the next point we show that also in the charts of the type (ii) the problem can be reduced to an analytic problem of the same type.

6. **Separating solutions of Pfaff systems.** In this point the equation $A(s;H) = 0$ (briefly $A(s) = 0$) is treated as an equation for s depending on the parameter H. Also the variables z_i are treated as functions of s. The exterior derivative, denoted by d, means the derivative with respect to v,u,s with H fixed.

The equation $A(s) = 0$ can be rewritten in the form

$$A_0|_\Gamma = 0,$$

where Γ is a separating solution of the following Pfaff system (of M equations),

$$\begin{array}{rcl} z_i dv_i & = & v_i^2 dz_i, \\ z_i du_{ij} & = & \alpha_{ij} u_{ij} dz_i, \end{array} \tag{2.4}$$

(recall that $A(s) = A_0(z(s),v(s),u(s))$).

According to Khovanski, a **separating solution** of a Pfaff equation $\eta = 0$ in a real manifold U (where η is a 1–form) is a smooth hypersurface Γ (with the embedding $i : \Gamma \to U$) such that:

(a) $i^*\eta \equiv 0$ (i.e. Γ is an integral surface of the Pfaff equation);

(b) Γ does not pass through singular points of η (i.e. $\eta(x) \neq 0$ for $x \in \Gamma$) and

(c) Γ is the boundary of a domain $D \subset U$ and the coorientation of Γ by means of η coincides with its coorientation as the boundary ∂D.

A smooth submanifold $\Gamma \subset U$ of codimension l is a **separating solution** of a Pfaff system $\eta_1 = \ldots = \eta_l = 0$ iff there is a sequence of submanifolds $\Gamma = \Gamma_0 \subset \Gamma_1 \subset \ldots \subset \Gamma_l = U$ such that each Γ_{j-1} is a separating solution of $\eta_j|_{\Gamma_j}$ in Γ_j.
There holds the following fundamental statement:

Khovanski–Rolle lemma. *Let $\Gamma \subset U$ be a separating solution of a Pfaff equation $\eta = 0$ and let $\gamma \subset U$ be an oriented curve. Then between any two consecutive intersection points of γ with Γ there exists a point (in γ), where the tangent vector γ_* (of γ) lies in the hyperplane $\eta = 0$.*

Proof. Assume that the intersections are transversal; (if not, then $\langle \eta, \gamma_* \rangle = 0$ at one of the intersection points. In the consecutive points of $\gamma \cap \Gamma$ the values of η at γ_* have different signs. This implies that $\langle \eta, \gamma_* \rangle$ vanishes at some intermediary point. □

The Khovanski–Rolle lemma allows us to replace the non-analytic equation $A(s) = 0$ by a collection of systems of analytic equations of the type $A_0(v,u,s) = A_1(v,u,s) = \ldots = A_M(v,u,s) = 0$ in $U = (0,1)^M \times \{t : z_i(t) \in (0,1),\ i = 1,\ldots,N\}$, depending analytically on H.
Indeed, the formulas $v_i = (-\ln z_i)^{-1}$, $u_{ij} = z_i^{\alpha_{ij}}$ define a 1-dimensional submanifold $\Gamma_0 \subset U$. It is a separating solution of the Pfaff system (2.4). To see this, it is enough to introduce some order in the system (2.4) and observe that each hypersurface $v_i = (-\ln z_i)^{-1}$ (or $u_{ij} = z_i^{\alpha_{ij}}$) is a boundary. Thus we have the sequence $\Gamma_0 \subset \Gamma_1 \subset \ldots \subset \Gamma_M = U$.
Take the surface Γ_1. It contains the separating solution Γ_0 of the Pfaff equation $\eta_1|_{\Gamma_1} = 0$ and the curve $\gamma_0 = \{A_0|_{\Gamma_1} = 0\}$. We are interested in estimation of the number (multiplicity counting) of intersection points $\gamma_0 \cap \Gamma_0$. On each compact connected component of γ_0 this number is equal to the number of contact points of this component with the field of directions $\eta_1 = 0$ (by the Khovanski–Rolle lemma). On each non-compact connected component of γ_0 this number is equal to 1 plus the number of contact points of this component with the field of directions $\eta_1 = 0$. The contact points are defined by the system of analytic equations in Γ_1: $A_0 = 0$, $dA_0 \wedge \eta_1 = 0$ (or $A_0 = A_1 = 0$). We find that

$$\#\{A = 0\} = \#\{\text{non-compact components of } \gamma_0\} + \#\{A_0 = A_1 = 0\}.$$

The number of non-compact components can be estimated as follows. Following [**Kh2**] we introduce the *special bump function* $\chi(s,v,u)$ which is positive, analytic and vanishes at the boundary of U, e.g. $\chi = \prod z_i(1-z_i) \prod v_i(1-v_i) \prod u_{ij}(1-u_{ij})$. Take a small positive value ϵ_1, non-critical for χ. We have $\#(\Gamma_1 \cap \{\chi - \epsilon_1 = 0\}) \geq 2 \cdot \#\{\text{non-compact components}\}$.

Consider the 3-dimensional manifold Γ_2. It contains the separating solution Γ_1 of $\eta_2 = 0$ and the curves $A_0 = A_1 = 0$ and $A_0 = \chi - \epsilon_1 = 0$. As in the previous case we obtain that $\#\Gamma_1 \cap$ {curve}$=$ #{non-compact components of curve}+#{contact points of curve}. The number of contact points leads to a system of three analytic equations. The number of non-compact components is estimated using the special bump function χ and some non-critical value ϵ_2.
Repeating this algorithm a finite number of times we reduce the problem of estimation of the number of solutions of $A(s) = 0$ to the problem of estimations of the numbers of solutions of 2^M systems of analytic equations in U.
Recall also that the estimate should be locally uniform with respect to the parameter H, i.e. for $H \in V = V_i \subset \mathcal{H}$. It may occur (and occurs) that for some H the system has infinitely many solutions, the set of solutions becomes a variety of positive dimension. Because (in the Hilbert's problem) we are interested in the isolated periodic trajectories of a vector field, here we are interested in the *isolated* solutions of the systems of analytic equations.
We can formulate the problem as follows. Consider a real semi-analytic set $W = \{((z,v,u);H) : (z,v,u) \in U,\ H \in V,\ z = z(s,H),\ A_0(z,v,u) = A_1 = \ldots = A_M = 0\} \subset U \times V$. We have the projection $\Pi : W \to V$, the restriction of the projection onto the second factor. The problem is:

Show that there exists a constant C such that the number of isolated points in $\Pi^{-1}(H)$ is $\leq C$ for any $H \in V$.

The following result completes the proof of Theorem 6.26. □

6.27. Theorem of Gabrielov. ([**Gab**]) *Such a constant C exists.*

6.28. Remarks. This theorem is a result from real analytic geometry. Recall that by definition a subset $W \subset \mathbb{R}^m$ is (real) **analytic** iff near any point $x_0 \in \mathbb{R}^n$ (not only in W!) it is defined by a system of equations analytic in a neighborhood of x_0. W is (real) **semi-analytic** iff near any point $x_0 \in \mathbb{R}^n$ it is a finite union of subsets defined by finite systems of analytic equations and analytic inequalities. Thus the above sets U, V, W are real semi-analytic. When the equations (or/and inequalities) are algebraic, then we have *real (semi-)algebraic sets.* The real semi-algebraic sets form real analogies of the complex quasi-projective varieties (i.e. Zariski open subsets of projective algebraic varieties) and of the complex constructible sets (i.e. finite unions of sets which are Zariski open in their closures).
The complex analytic, algebraic and quasi-projective varieties have some nice natural properties. A complex analytic variety has locally finitely many components. The image $f(W)$ of a complex analytic variety $W \subset \mathbb{C}^n$, under a *proper* analytic map f, is an analytic set (theorem of Remmert, see [**GH**]). The equivalent formulation of this statement says that a proper projection of a complex analytic set is an analytic set. Here the assumption that f is proper (i.e. that the inverse images of compact subsets are compact) is essential: for example, the image of the hyperbola $xy = 1$ under the projection onto the y-axis equals $\mathbb{C}^*$. Also the existence of

a local uniform bound for the number of connected components of $f^{-1}(y)$ is easy in the complex case.
In the real case the situation is not that clear. Real (semi-)analytic subsets $W \subset \mathbb{R}^n$ have finitely many local (e.g. in a cube) connected components (see [**Loj1**]). In fact, one can prove this property following the lines of the proof of Lemma 4.2 in Chapter 4. It was shown there that if a real algebraic variety (or real analytic variety) has a sequence of points accumulating at x_0, then it contains a semi-analytic curve through x_0. Real semi-analytic sets can be stratified; they form CW-complexes with some additional differential properties.
A. Seidenberg and A. Tarski proved that a projection (or equivalently an image under a real algebraic map) of a semi-algebraic set is semi-algebraic (see [**Loj1**]). In the proof they used methods from mathematical logic, but there is an analytic proof based on algebraic functions.
The Bezout theorem (about estimation of the number of solutions of a system of complex algebraic equations by the product of degrees) does not hold in the real case; here is the example:

$$x = y = x^2 + y^2 + [z(z^2-1)]^2 + [t(t^2-1)]^2 = 0.$$

The example with the analytic set

$$\begin{aligned} W &= \{(x,y,z,u,s,t) : x^2+y^2+z^2+u^2 = 1,\ s^2+t^2 = 1, \\ &\qquad z = x \cdot \exp\left[s^2/(s^2+2t^2)\right], tx = sy\} \subset \mathbb{R}^4 \times \mathbb{R}^2 \end{aligned}$$

with its projection f onto $\mathbb{R}^4$, equal to $f(W) = \{z = x \cdot \exp\left[x^2/(x^2+2y^2)\right],$ $x^2+y^2+z^2+u^2 = 1\}$ (which is not analytic at $(0,0,0,1)$), shows that proper projections of semi-analytic sets can be not semi-analytic. This may occur only when $\dim f(W) > 2$. The subsets of $\mathbb{R}^n$, which are projections of relatively compact semi-analytic sets are called the *$\mathcal{P}$-sets,* or the *sub-analytic sets.* Here the assumption of relative compactness replaces the assumption of properness. It is useful here to have in mind the following example, with the (not relatively compact) analytic set $W = \{(x,y,z) : z(x^2+y^2) = 1,\ |x+iy| = \arg(x+iy)\} \subset \mathbb{R}^3 = \mathbb{R}^2 \times \mathbb{R}$ whose projection is the logarithmic spiral.
The sub-analytic sets have the following properties. Their complements are sub-analytic sets. Near any point they are defined by a system of analytic equations and inequalities. They are locally arcwise connected (by means of semi-analytic curves). There holds a sub-analytic analogue of Gabrielov's theorem (it is in fact the original theorem of Gabrielov): if $Z \subset [0,1]^m \times [0,1]^n$ is a sub-analytic set and Π is the restriction (to Z) of the projection onto the second factor, then the numbers of connected components of the fibers $\Pi^{-1}(y)$ are uniformly bounded.

6.29. Sketch of the proof of Gabrielov's theorem. Because the original proof of Gabrielov concerns the general case of projection of a $\mathcal{P}$-set and is rather technical, we provide an independent proof. We will deal with restricted assumptions (for simplicity):

We have an analytic subset W of an open bounded domain $U \subset \mathbb{R}^{m+n} = \{(x,y)\}$ such that the restriction Π of the projection onto $\mathbb{R}^n = \{y\}$ has fibers either finite or $\geq$ 1-dimensional, but the typical fiber is finite (maybe empty). We assume also that the variable x is 1-dimensional (i.e. $m = 1$); later we will say how we work in the general case. We denote the fibers by $W_y = \Pi^{-1}(y)$.

Assume that the thesis of Gabrielov's theorem does not hold. It means that there exists a sequence of points $y_k \to y_*$ (which we put $= 0$) such that the fibers W_{y_k} contain a growing number of bounded isolated points x_{kj}. Assume that a point (x_*, y_*) (which we put $= (0,0)$) is an accumulation point of the set $\{(x_{kj}, y_k)\}$.

Let $f_1(x,y), \ldots, f_r(x,y)$ be the generators of the ideal (in the local ring $\mathcal{O}_0(\mathbb{R}^{n+1})$) of germs of functions vanishing at W. If some of the functions f_j is such that its restriction to the line $y = 0$ is of finite multiplicity, $f_j(x,0) = ax^d + \ldots$, then any fiber W_y does not contain more than d points (by the Weierstrass theorem).

Assume then that all $f_j(x,0) \equiv 0$; in other words the line $\{y = 0\} \subset W$. Here the essence of the theorem lies. When we treat f_j as functions of x depending on the parameter y, then for $y = 0$ the point $x = 0$ is singular of infinite codimension. In the smooth (i.e. C^∞) case an unbounded number of singular points could be born after perturbation, i.e. for $y \neq 0$. In the analytic case this number turns out to be bounded. We shall meet this phenomenon in the next section (see also [**FY**]).

Let $Z = \Pi(W) \subset \mathbb{R}^n$. It is a sub-analytic set (maybe not semi-analytic) containing the origin $y = 0$. We have a mapping $\Pi : W \to Z$ between sets of the same dimension. The cardinality of the fiber changes. The changes of the cardinality of $\Pi^{-1}(y)$ occur due to bifurcations. Such bifurcation points are the critical points of the restriction of the projection Π to the corresponding stratum of the stratification of W (the fold singularity).

These critical points are obtained by differentiation of the generators f_j with respect to x. Indeed, if (locally outside $y = 0$) W is represented as $F(x;y) = 0$, $F : \mathbb{R} \times Z \to \mathbb{R}$, then the Rolle principle says that we should calculate the zeroes of $\partial_x F$.

The system of equations $F = \partial_x F = 0$ defines a subvariety $W_1 \subset W$, which can be defined in an analytic way. W_1 is an analytic variety. It contains the line $y = 0$. Let $Z_1 = \Pi(W_1) \subset \mathbb{R}^n$ and $\widetilde{W_1} = \Pi^{-1}(Z_1) \subset W$. The latter are sub-analytic sets (in general). The restriction $\Pi|_{\widetilde{W_1}} : \widetilde{W_1} \to Z_1$ is a mapping with the same properties as $\Pi : W \to Z$. The cardinalities of fibers are unbounded.

We proceed as before, we look for the critical points of Π_1. It is not difficult to see that they form an analytic subset $W_2 \subset W_1$, defined by means of $f_j, \partial_x f_j$ and $\partial_x^2 f_j$. We put $Z_2 = \Pi(W_2)$, $\widetilde{W_2} = \Pi^{-1}(Z_2)$. The mapping $\Pi|_{\widetilde{W_2}}$ has the same properties as $\Pi|_{\widetilde{W_1}}$ etc.

Due to the analyticity of W, the sequence of varieties W_j stabilizes, $W_p = W_{p+1} = \ldots = W_\infty$ (because of the dimension argument); assume that p is a minimal such integer. The generators of the ideal of functions vanishing at W_∞ have the property that all their derivatives with respect to x are equal to zero; they depend only on y. Thus $W_\infty = \mathbb{R} \times Z_\infty$.

For any $y \in Z - Z_\infty$ the maximal order of zero of the function $F(\cdot, y)$ (where $W = \{F = 0\}$) is $p - 1$. Thus $\partial_x^{p-1} F \neq 0$ near $x = 0$ and there are no more than $p - 1$ isolated zeroes of F.
For example, if $n = 2$ and $W = \{F = y_1 g_1(x, y) + y_2 g_2(x, y) = 0\}$ then we get $W_1 = \{y = 0\} \cup \{F = \Delta = 0\}$, $\Delta = g_1 g_{2x} - g_2 g_{1x}$. Assume that the surface $\Delta = 0$ intersects the line $y = 0$ transversally; then we have $W_\infty = \{y = 0\}$ and the curve $\{F = \Delta = 0\}$ is the curve of non-degenerate fold points. Any fiber W_y, $y \neq 0$ contains at most two points.

The general case of multi-dimensional $x = (x_1, \ldots, x_m)$ can be treated (in principle) in the same way. We take the generators $f_1, \ldots, f_r$, $r \geq m$ of the ideal of functions vanishing on W. If the restrictions to the plane $y = 0$ of some m of the f_j's form a vector field of finite multiplicity, then we get the bound for cardinalities of the fibers. Otherwise we look for the critical points of the map $\Pi : W \to Z = \Pi(W)$. These critical points form an analytic subset W_1, etc. □

§3 Quadratic Centers and Bautin's Theorem

The correspondence between zeroes of Abelian integrals and limit cycles of perturbations of Hamiltonian systems is not one-to-one. In particular, existence of a uniform bound for the number of zeroes of Abelian integrals does not lead automatically to a uniform bound for the number of limit cycles. As an example, where there can be more limit cycles than zeroes of the Abelian integrals, we consider the case of quadratic perturbation of the linear center.

Consider the perturbation

$$\dot{x} = -y + \epsilon P, \ \dot{y} = x + \epsilon Q,$$

where P and Q are quadratic polynomials. Without loss of generality we can assume that the point $(0, 0)$ remains fixed during the perturbation and that the linear part has the canonical form with the complex eigenvalues $\lambda \pm i, \lambda =$const$\cdot\epsilon$. Then the corresponding Abelian integral reduces to

$$\epsilon I = \lambda \int y dx - x dy = 2\lambda \times \text{area}$$

and, of course, $I \neq 0$. However we have the following result.

6.30. Theorem of Bautin. ([**Baut**]) *The maximal number of limit cycles appearing after the above perturbation is equal to* 3.

Proof. In order to calculate the number of all limit cycles we have to calculate the Poincaré return map up to a sufficiently high exactness

$$\Delta H = \epsilon I + \epsilon^2 I_2 + \ldots.$$