$\exp \mathbf{K}$, where $\mathbf{K} = \mathbf{K}(\mathbf{F}, \mathbf{G})$ is a series which in the nilpotent case is a polynomial map $\mathfrak{t} \times \mathfrak{t} \to \mathfrak{t}$ defined everywhere. Thus the image $\exp \mathfrak{g}$ is a *Lie subgroup* in $\mathfrak{G} \subseteq \mathfrak{T}$ for *any* subalgebra $\mathfrak{g}$, containing a small neighborhood of the unit $\mathbf{E}$. It is well known that any such neighborhood generates (by the group operation) the whole connected component of the unit, so that $\exp \mathfrak{g}$ coincides with this component. If $\mathfrak{G}$ is simply connected, then $\exp \mathfrak{g} = \mathfrak{G}$ as asserted.

Without nilpotency the answer may be different: as follows from Remark 3.13, for two Lie algebras $\mathfrak{gl}(n, \mathbb{R}) \subset \mathfrak{gl}(n, \mathbb{C})$ and the respective Lie groups $\mathrm{GL}(n, \mathbb{R}) \subset \mathrm{GL}(n, \mathbb{C})$, the exponent is surjective on the ambient (bigger) group but *not* on the subgroup.

**Remark 3.16.** Using similar arguments, one can prove that for an arbitrary automorphism $\mathbf{H} \in \mathrm{Aut}(\mathfrak{F})$ *sufficiently close to the unit* $\mathbf{E}$, the logarithm $\ln \mathbf{H}$ given by the series (3.12) is a derivation, $\ln \mathbf{H} \in \mathrm{Der}(\mathfrak{F})$. This follows from the fact that ln and exp are mutually inverse on sufficiently small neighborhoods of $\mathbf{E}$ and 0 respectively. However, the size of this neighborhood depends on $\mathfrak{F}$.

**3F. Embedding in the formal flow.** Based on Theorem 3.14, one can prove the following general result obtained by F. Takens in 1974; see [**Tak01**].

**Theorem 3.17.** *Let $H \in \mathrm{Diff}[[\mathbb{C}^n, 0]]$ be a formal map whose linearization matrix $A = \frac{\partial H}{\partial x}(0)$ is unipotent, $(A - E)^n = 0$.*

*Then there exists a formal vector field $F \in \mathcal{D}[[\mathbb{C}^n, 0]]$ whose linearization is a nilpotent matrix $N$, such that $H$ is the formal time 1 map of $F$.*

**Proof.** As usual, we identify the formal map with an automorphism $\mathbf{H}$ of the algebra $\mathfrak{F} = \mathbb{C}[[x_1, \dots, x_n]]$ so that its finite $k$-jets $j^k\mathbf{H}$ become automorphisms of the finite dimensional algebras $\mathfrak{F}^k = J^k(\mathbb{C}^n, 0)$. Without loss of generality we may assume that the matrix $A$ is upper-triangular so that the isomorphism $\mathbf{H}$ and all its truncations $j^k\mathbf{H}$ in the canonical `deglex`-ordered basis becomes upper-triangular with units on the diagonal: the jets $j^k\mathbf{H}$ are finite-dimensional upper-triangular (unipotent) automorphisms of the algebras $\mathfrak{F}^k$.

Consider the infinite series (3.12) together with its finite-dimensional truncations obtained by applying the operation $j^k$ to all terms. Each such truncation is a logarithmic series for $\ln j^k\mathbf{H}$ which converges (actually, stabilizes after finitely many steps) and its sum is a derivation $j^k\mathbf{F}$ of $\mathfrak{F}^k$ by Theorem 3.14. Clearly, different truncations agree on the lower order terms, thus $\ln \mathbf{H}$ converges in the sense of Definition 3.4 to a derivation $\mathbf{F}$ of $\mathfrak{F}$. This derivation corresponds to the formal vector field $F$ as required. □

### Exercises and Problems for §3.

**Problem 3.1.** Let $F \in \mathcal{D}[[\mathbb{C}^n, 0]]$ be a formal vector field corresponding to the derivation $\mathbf{F} \in \operatorname{Der} \mathbb{C}[[x]]$, and $\{H^t\} \subset \operatorname{Diff}[[\mathbb{C}^n, 0]]$ its formal flow corresponding to the one-parametric group of automorphisms $\{\mathbf{H}^t\} \subset \operatorname{Aut} \mathbb{C}[[x]]$ related by the identity (3.7).

Prove that in this case $\frac{d}{dt} H^t = F \circ H^t$ for any $t$ on the level of vector formal series.

**Exercise 3.2.** Consider the derivation $\mathbf{F} = \frac{\partial}{\partial x}$ on the algebra $\mathbb{C}[x]$ of univariate polynomials. Prove that the exponential series $\exp t\mathbf{F}$ is well defined for all values of $t \in \mathbb{C}$ as an automorphism of $\mathbb{C}[x]$, but is not defined if the algebra $\mathbb{C}[x]$ is replaced by the algebras $\mathbb{C}[[x]]$ or $\mathcal{O}(\mathbb{D})$, where $\mathbb{D} = \{|x| < 1\}$ is the unit disk.

**Problem 3.3.** Prove that the integral representation (3.11) coincides with the standard definition of a matrix function $f(M)$ in the case where $f$ is a (scalar) polynomial.

**Exercise 3.4.** Find *all* complex logarithms of the matrix $M = \begin{pmatrix} -1 & 1 \\ & -1 \end{pmatrix}$ (i.e., solutions of the equation $\exp A = M$).

## 4. Formal normal forms

In the same way as holomorphic maps act on holomorphic vector fields by conjugacy (1.26), formal maps act on formal vector fields. In this section we investigate the *formal normal forms*, to which a formal vector field can be brought by a suitable formal isomorphism.

**Definition 4.1.** Two formal vector fields $F, F'$ are *formally equivalent*, if there exists an invertible formal self-map $H$ such that the identity (1.26) holds on the level of formal series.

The fields are formally equivalent if and only if the corresponding derivations $\mathbf{F}, \mathbf{F}'$ of the algebra $\mathbb{C}[[x]]$ are conjugated by a suitable isomorphism $\mathbf{H} \in \operatorname{Diff}[[\mathbb{C}^n, 0]]$ of the formal algebra: $\mathbf{H} \circ \mathbf{F}' = \mathbf{F} \circ \mathbf{H}$.

Obviously, two holomorphically equivalent (holomorphic) germs of vector fields are formally equivalent. The converse is in general not true, as the formal self-maps may be divergent.

**4A. Formal classification theorem.** Formal classification of formal vector fields strongly depends on its principal part, in particular, on properties of the linearization matrix $A = \left(\frac{\partial F}{\partial x}\right)(0)$ when the latter is nonzero (cases with $A = 0$ are hopelessly complicated if the dimension is greater than one).

We start with the most important example and introduce the definition of a resonance as a certain arithmetic (i.e., involving integer coefficients) relation between complex numbers.

**Definition 4.2.** An ordered tuple of complex numbers $\lambda = (\lambda_1, \dots, \lambda_n) \in \mathbb{C}^n$ is called *resonant*, or, more precisely, *additive resonance* if there exist nonnegative integers $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{Z}_+^n$ such that $|\alpha| \geqslant 2$ and the *resonance* identity occurs,

$$\lambda_j = \langle \alpha, \lambda \rangle, \qquad |\alpha| \geqslant 2. \tag{4.1}$$

Here $\langle \alpha, \lambda \rangle = \alpha_1\lambda_1 + \cdots + \alpha_n\lambda_n$. The natural number $|\alpha|$ is the *order* of the resonance.

A square matrix is resonant, if the collection of its eigenvalues (with repetitions if they are multiple) is resonant. A formal vector field $F = (F_1, \dots, F_n)$ at the origin is resonant if its linearization matrix $A = \left(\frac{\partial F}{\partial x}\right)(0)$ is resonant.

Though resonant tuples $(\lambda_1, \dots, \lambda_n)$ can be dense in some parts of $\mathbb{C}^n$ (see §5**A**), their measure is zero.

**Theorem 4.3** (Poincaré linearization theorem)**.** *A nonresonant formal vector field* $F(x) = Ax + \cdots$ *is formally equivalent to its linearization* $F'(x) = Ax$.

The proof of this theorem is given in the sections §4**B**–§4**C**. In fact, it is the simplest particular case of a more general statement valid for resonant formal vector fields that appears in §4**D**.

**4B. Induction step: homological equation.** The proof of Theorem 4.3 goes by induction. Assume that the formal vector field $F$ is already partially normalized, and contains no terms of order less than some $m \geqslant 2$:

$$F(x) = Ax + V_m(x) + V_{m+1}(x) + \cdots,$$

where $V_m, V_{m+1}, \dots$ are arbitrary homogeneous vector fields of degrees $m, m+1$, *etc.*

We show that in the assumptions of the Poincaré theorem, the term $V_m$ can be removed from the expansion of $F$, i.e., that $F$ is formally equivalent to the formal field $F'(x) = Ax + V'_{m+1} + \cdots$. Moreover, the corresponding conjugacy can be in fact chosen as a polynomial of the form $H(x) = x + P_m(x)$, where $P_m$ is a homogeneous vector polynomial of degree $m$. The Jacobian matrix of this self-map is $E + \left(\frac{\partial P_m}{\partial x}\right)$.

The conjugacy $H$ with these properties must satisfy the equation (1.26) on the formal level. Keeping only terms of order $\leqslant m$ from this equation and using dots to denote the rest, we obtain

$$\left(E + \frac{\partial P_m}{\partial x}\right)(Ax + V_m + \cdots) = A(x + P_m(x)) + V'_m(x + P_m(x)) + \cdots.$$

The homogeneous terms of order 1 on both sides coincide. The next non-trivial terms appear in the order $m$. Collecting them, we see that in order to meet the condition $V'_m = 0$, the vector of homogeneous terms $P_m$ must satisfy the commutator identity

$$[\mathbf{A}, P_m] = -V_m, \qquad \mathbf{A}(x) = Ax, \tag{4.2}$$

where $\mathbf{A} = Ax$ is the linear vector field, the principal part of $F$, and the homogeneous vector polynomials $P_m$ and $V_m$ are considered as vector fields on $\mathbb{C}^n$. The left hand side of (4.2) is the commutator, $[\mathbf{A}, P](x) = \left(\frac{\partial P}{\partial x}\right) Ax - AP(x)$.

Conversely, if the condition (4.2) is satisfied by $P_m$, the polynomial map $H(x) = x + P_m(x)$ conjugates $F = \mathbf{A} + V_m + \cdots$ with the (formal) vector field $F'(x) = \mathbf{A} + \cdots$ having no terms of degree $m$.

**Definition 4.4.** The identity (4.2), considered as an equation on the unknown homogeneous vector field $P_m$, is called the *homological equation.*

**4C. Solvability of the homological equation.** Solvability of the homological equation depends on invertibility of the operator $\operatorname{ad}_A$ of commutation with the linear vector field $\mathbf{A}$.

Let $\mathcal{D}_m$ be the linear space of all homogeneous vector fields of degree $m$ (we will be interested only in the case $m \geqslant 2$). This linear space has the *standard monomial basis* consisting of the fields

$$F_{k\alpha} = x^\alpha \tfrac{\partial}{\partial x_k}, \qquad k = 1, \dots, n, \ |\alpha| = m. \tag{4.3}$$

We shall order elements of this basis lexicographically so that $x_i$ precedes $x_j$ if $i < j$, but $\frac{\partial}{\partial x_j}$ precedes $\frac{\partial}{\partial x_i}$. To that end, we assign to each formal variable $x_1, \dots, x_n$ pairwise different positive weights $w_1 > \cdots > w_n$ that are *rationally independent.* This assignment extends on all monomials and monomial vector fields if the symbol $\frac{\partial}{\partial x_j}$ is assigned the weight $-w_j$. Now the monomial vector fields can be arranged in the decreasing order of their weights: the independence condition guarantees that the only different vector monomials having the same weight can be $x^\alpha \cdot x_j \frac{\partial}{\partial x_j}$ and $x^\alpha \cdot x_k \frac{\partial}{\partial x_k}$ with the same $\alpha$ and $j \neq k$. The order between these monomials is not essential for future exposition.

The operator

$$\operatorname{ad}_A \colon P \mapsto [\mathbf{A}, P], \qquad (\operatorname{ad}_A P)(x) = \left(\frac{\partial P}{\partial x}\right) \cdot Ax - AP(x), \tag{4.4}$$

preserves the space $\mathcal{D}_m$ for any $m \in \mathbb{N}$.

**Lemma 4.5.** *If $A$ is nonresonant, then the operator* $\operatorname{ad}_A$ *is invertible. More precisely, if the coordinates $x_1, \dots, x_n$ are chosen such that $A$ has the upper-triangular Jordan form, then* $\operatorname{ad}_A$ *is lower-triangular in the respective standard monomial basis ordered in the decreasing weight order.*

**Proof.** The assertion of the lemma is completely transparent when $A$ is a diagonal matrix $\Lambda = \operatorname{diag}\{\lambda_1, \dots, \lambda_n\}$. In this case each $F_{k\alpha} \in \mathcal{D}_m$ is an eigenvector for $\operatorname{ad}_\Lambda$ with the eigenvalue $\langle \lambda, \alpha \rangle - \lambda_k$. Indeed, by the Euler identity,

$$F_{k\alpha} = x^\alpha \begin{pmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix}, \qquad \left(\frac{\partial F_{k\alpha}}{\partial x}\right) = x^\alpha \begin{pmatrix} & 0 & \\ & \vdots & \\ \frac{\alpha_1}{x_1} & \cdots & \frac{\alpha_n}{x_n} \\ & \vdots & \\ & 0 & \end{pmatrix},$$

so that in the diagonal case $\Lambda F_{k\alpha} = \lambda_k F_{k\alpha}$, and $\left(\frac{\partial F_{k\alpha}}{\partial x}\right) \Lambda x = \langle \lambda, \alpha \rangle F_{k\alpha}$. Being diagonal with nonzero eigenvalues, $\operatorname{ad}_\Lambda$ is invertible.

To prove the lemma in the general case where $A$ is in the upper-triangular Jordan form, we consider the weight introduced above.

The operator $\operatorname{ad}_\Lambda$ with the diagonal matrix $\Lambda$ preserves the weights, since all vector monomials are eigenvectors for it.

On the other hand, the monomial vector field $\mathbf{J}_j = x_j \frac{\partial}{\partial x_{j+1}}$ with the upper-diagonal constant matrix $J_j$ acts by increasing weight. Indeed,

$$\left[x^\alpha \frac{\partial}{\partial x_k}, x_j \frac{\partial}{\partial x_{j+1}}\right] = x^\alpha \left[\frac{\partial}{\partial x_k}, x_j \frac{\partial}{\partial x_{j+1}}\right] + \alpha_{j+1} x^\alpha \frac{x_j}{x_{j+1}} \cdot \frac{\partial}{\partial x_k}.$$

The second term, if present, has higher weight than $F_{k\alpha} = x^\alpha \frac{\partial}{\partial x_k}$, since $w_j > w_{j+1}$. The first term is nonzero only if $j = k$, and in this case reduces to $x^\alpha \frac{\partial}{\partial x_{k+1}}$, which also has higher weight than $F_{k\alpha}$.

It remains to notice that an arbitrary matrix $A$ in the upper-triangular Jordan normal form is the sum of the diagonal part $\Lambda$ and a linear combination of matrices $J_1, \dots, J_{n-1}$. The operator $\operatorname{ad}_A$ linearly depends on $A$, so $\operatorname{ad}_A$ is equal to $\operatorname{ad}_\Lambda$ modulo a linear combination of the weight-increasing operators $\operatorname{ad}_{J_j}$. Therefore, if the monomial fields $F_{k\alpha}$ are ordered in the decreasing order of their weights, as in the standard basis, then the operator $\operatorname{ad}_A$ is lower-triangular with the diagonal part $\operatorname{ad}_\Lambda$. □

**Proof of Theorem 4.3.** Now we can prove the Poincaré linearization theorem. By Lemma 4.5, the operator $\operatorname{ad}_A$ is invertible and therefore the homological equation (4.2) is always solvable no matter what the term

$V = V_m$ is. Repeating this process inductively, we can construct an infinite sequence of polynomial maps $H_1, H_2, \ldots, H_m, \ldots$ and the formal fields $F_1 = F$, $F_2, \ldots, F_m, \ldots$ such that $F_m = Ax +$ (terms of order $m$ and more), and the transformation $H_m = \mathrm{id} +$(terms of order $m$ and more) conjugates $F_m$ with $F_{m+1}$. Thus the composition $H^{(m)} = H_m \circ \cdots \circ H_1$ conjugates the initial field $F_1$ with the field $F_{m+1}$ without nonlinear terms up to order $m$.

It remains to notice that by construction of $H_{m+1}$ the composition $H^{(m+1)} = H_{m+1} \circ H^{(m)}$ has the same terms of order $\leqslant m$ as $H^{(m)}$ itself. Thus the limit

$$H = H^{(\infty)} = \lim_{m\to\infty} H^{(m)}$$

(the infinite composition) exists in the class of formal morphisms. By construction, $H_*F$ cannot contain any nonlinear terms and hence is linear, as required. □

**Remark 4.6.** The formal map linearizing a nonresonant formal vector field and tangent to the identity, is unique. Indeed, otherwise there would exist a *nontrivial* formal map $\mathrm{id} + h$ which conjugates the linear field with itself,

$$\left(\frac{\partial h}{\partial x}\right) Ax = Ah(x), \qquad \text{i.e.,} \qquad \mathrm{ad}_A\, h = 0.$$

But in the nonresonant case the commutator $\mathrm{ad}_A$ is injective, hence $h = 0$.

Thus the only formal maps conjugating a linear field with itself, are linear maps $x \mapsto Bx$, with the matrix $B$ commuting with $A$, $[A, B] = 0$.

**4D. Resonant normal forms: Poincaré–Dulac paradigm.** The inductive construction linearizing nonresonant vector fields, can be used to *simplify* the resonant ones.

In this *resonant* case the operator $\mathrm{ad}_A = [\mathbf{A}, \cdot]$ of commutation with the linear part may be no longer surjective and in general the condition $V'_m = 0$, meaning absence of terms of order $m$ after the transformation, cannot be achieved.

In the presence of resonances one can choose in each linear space $\mathcal{D}_m$ a complementary (transversal) subspace $\mathcal{N}_m$ to the image of the operator $\mathrm{ad}_A$, so that

$$\mathcal{D}_m = \mathcal{N}_m + \mathrm{ad}_A(\mathcal{D}_m) \tag{4.5}$$

(the sum should not necessarily be direct).

**Theorem 4.7** (Poincaré–Dulac paradigm). *If the subspaces $\mathcal{N}_m \subset \mathcal{D}_m$ are transversal to the image of the commutator $\mathrm{ad}_A$ as in* (4.5)*, then any formal vector field $F(x) = Ax + \cdots$ with the linearization matrix $A$ is formally conjugated to some formal vector field whose all nonlinear terms of degree $m$ belong to the subspace $\mathcal{N}_m$.*

**Proof.** If the transformation $H_m(x) = x + P_m$ conjugates the field $F(x) = Ax + \cdots + V_m(x) + \cdots$ with another field $F'(x) = Ax + \cdots + V'_m(x) + \cdots$ *with the same* $(m-1)$*-jet* on the level of terms of order $m$, then instead of the homological equation (4.2) in the case $V'_m \neq 0$, the correction term $P_m$ must satisfy the equation

$$\operatorname{ad}_A P_m = V'_m - V_m. \tag{4.6}$$

If $\mathcal{N}_m$ satisfies (4.5), then (4.6) can always be solved with respect to $P_m$ for any $V_m$ provided that $V'_m$ is suitably chosen from the subspace $\mathcal{N}_m$.

The transform $H_m$ does not affect the lower order terms and hence the process can be iterated for larger values of $m$ exactly as in the nonresonant case. As a result, one can prove that any formal vector field $F$ is formally equivalent to a formal field containing only terms belonging to the "complementary" parts $\mathcal{N}_m$ for all $m = 2, 3, \ldots$.

The rest of the proof of Theorem 4.7 is the same as that of the Poincaré–Dulac theorem. □

The choice of the transversal subspaces $\mathcal{N}_m$ depends on $\operatorname{ad}_A$, hence on the matrix $A$ itself.

**Example 4.8.** Assume that the matrix $A = \Lambda = \operatorname{diag}\{\lambda_1, \ldots, \lambda_n\}$ is diagonal. In this case the operator $\operatorname{ad}_\Lambda$ was already shown to be diagonal in the vector monomial basis, eventually with some zeros among the eigenvalues. For diagonal operators on finite-dimensional space the kernel and the image are complementary subspaces, so one may choose $\mathcal{N}_m = \ker \operatorname{ad}_L \subset \mathcal{D}_m$. The kernel of the diagonal operator $\operatorname{ad}_\Lambda$ can be immediately described.

**Definition 4.9.** A *resonant vector monomial* corresponding to the resonance $\lambda_k - \langle \lambda, \alpha \rangle = 0$, is the monomial vector field $F_{k\alpha} = x^\alpha \frac{\partial}{\partial x_k}$; see (4.3).

The kernel $\ker \operatorname{ad}_\Lambda$ consists of linear combinations of resonant monomials. From the discussion above it follows immediately that a formal vector field with diagonal linear part $\Lambda x$ is formally equivalent to the vector field with the same linear part and only resonant monomials among the nonlinear terms.

Actually, the assumption on diagonalizability is redundant. The following statement is one of the most popular formal classification results.

**Theorem 4.10** (Poincaré–Dulac theorem)**.** *A formal vector field is formally equivalent to a vector field with the linear part in the Jordan normal form and only resonant monomials in the nonlinear part.*

**Proof.** Assume that the coordinates are already chosen so that the linearization matrix $A$ is Jordan upper-triangular.

Choose the subspace $\mathcal{N}_m$ as the linear span of all resonant monomials, $\mathcal{N}_m = \bigoplus_{\langle\lambda,\alpha\rangle-\lambda_k=0} \mathbb{C}\cdot F_{k\alpha}$.

By Lemma 4.5, the operator $L_m = \operatorname{ad}_A|_{\mathcal{D}_m}$ is upper triangular with the expressions $\langle\lambda,\alpha\rangle - \lambda_k = 0$ on the diagonal. By the choice of $\mathcal{N}_m$, whenever zero occurs on the diagonal of $L$, the corresponding basis vector was included in $\mathcal{N}_m$. This obviously means (4.5). The rest is the Poincaré–Dulac paradigm. □

**4E. Belitskii theorem.** The choice of the "resonant normal form" (i.e., of the family of subspaces $\mathcal{N}_m$) in Theorem 4.10, is excessive in the sense that the *dimension* of these spaces (the number of parameters in the normal form) is not minimal. For example, if $A$ is a nonzero nilpotent Jordan matrix, then *all* monomials are resonant in the sense of Definition 4.9, whereas the image of $\operatorname{ad}_A$ is clearly nontrivial. We describe now one possible *minimal* choice, introduced by G. Belitskii [**Bel79**, Ch. II, §7].

Consider the standard Hermitian structure on the space $\mathbb{C}^n$, so that the basis vectors $e_j = \frac{\partial}{\partial x_j}$ form an orthonormal basis.

For any natural $m \geqslant 1$ denote by $\mathcal{H}_m$ the complex linear space of all homogeneous polynomials of degree $m$. We introduce the *standard Hermitian structure* in $\mathcal{H}_m$ in such a way that the normalized monomials $\varphi_\alpha = \frac{1}{\sqrt{\alpha!}}\,x^\alpha$ form an orthonormal basis,

$$(\varphi_\alpha, \varphi_\beta) = \delta_{\alpha\beta}, \qquad \varphi_\alpha = \frac{1}{\sqrt{\alpha!}}\,x^\alpha, \quad \alpha,\beta\in\mathbb{Z}_+^n,\ |\alpha| = |\beta| = m. \tag{4.7}$$

Here, as usual, $\alpha! = \alpha_1!\cdots\alpha_n!$ for $\alpha = (\alpha_1,\dots,\alpha_n)$, $0! = 1$ and $\delta_{\alpha\beta}$ is the standard Kronecker symbol.

Then the linear space $\mathcal{D}_m$ of homogeneous vector fields of degree $m$ can be naturally identified with the tensor product $\mathcal{D}_m = \mathcal{H}_m \otimes_{\mathbb{C}} \mathbb{C}^n$ and inherits the standard Hermitian structure for which the monomials $\varphi_\alpha \otimes e_k = \frac{1}{\sqrt{\alpha!}}\,F_{\alpha k}$ form an orthonormal basis.

Given a matrix $A \in \operatorname{Mat}(n,\mathbb{C})$, denote by $A^*$ the *adjoint matrix* obtained from $A$ by transposition and complex conjugacy: $a^*_{ij} = \bar a_{ji}$. If $\mathbf{A}(x) = Ax$ is the corresponding linear vector field on $\mathbb{C}^n$ and, respectively, $\mathbf{A}^*(x) = A^*x$, then both $\mathbf{A}, \mathbf{A}^*$ act as linear differential operators, $\mathbf{A} = \sum a_{ij}x_i\frac{\partial}{\partial x_j}$ and $\mathbf{A}^* = \sum \bar a_{ji}x_i\frac{\partial}{\partial x_j}$, on $\mathcal{H}_m$. Furthermore, the commutation operators $\operatorname{ad}_A = [\mathbf{A},\cdot]$ and $\operatorname{ad}_{A^*} = [\mathbf{A}^*,\cdot]$ are linear operators on $\mathcal{D}_m$.

The following statement claims that the operators in each pair are mutually adjoint (dual to each other) with respect to the standard Hermitian structures on the respective spaces.

**Lemma 4.11.**

*1. The derivation* $\mathbf{A}^*\colon \mathcal{H}_m \to \mathcal{H}_m$ *is adjoint to the derivation* $\mathbf{A}$ (*with respect to the standard Hermitian structure*) *and vice versa.*

*2. The commutator* $\operatorname{ad}_{A^*} = [\mathbf{A}^*, \cdot]\colon \mathcal{D}_m \to \mathcal{D}_m$ *is adjoint to the commutator* $\operatorname{ad}_A = [\mathbf{A}, \cdot]$ (*with respect to the standard Hermitian structure*) *and vice versa.*

**Proof.** 1. The identity $(\mathbf{A}f, g) = (f, \mathbf{A}^*g)$ for any pair of polynomials $f, g \in \mathcal{H}_m$ "linearly" depends on the matrix $A$: if it holds for two matrices $A, A' \in \operatorname{Mat}(n, \mathbb{C})$, then it also holds for their combination $\lambda A + \lambda' A'$ with any two complex numbers $\lambda, \lambda' \in \mathbb{C}$.

Thus it is sufficient to verify the assertion for the monomial derivations $\mathbf{A} = x_i \frac{\partial}{\partial x_j}$ and $\mathbf{A}^* = x_j \frac{\partial}{\partial x_i}$.

If $i = j$, then $\mathbf{A} = \mathbf{A}^* = x_i \frac{\partial}{\partial x_i}$ is diagonal in the orthonormal basis $\{\varphi_\alpha\}$ with the real eigenvalues $\lambda_\alpha = \alpha_i = \alpha_j \in \mathbf{Z}_+$, and hence is self-adjoint.

Otherwise both $\mathbf{A}$ and $\mathbf{A}^*$ can be represented as permutations of the basic vectors composed with the diagonal operators. If $\beta$ is the multi-index obtained from $\alpha$ by the operation

$$\beta_k = \begin{cases} \alpha_k, & k \ne i, j, \\ \alpha_i + 1, & k = i, \\ \alpha_j - 1, & k = j, \end{cases} \qquad \alpha_k = \begin{cases} \beta_k, & k \ne i, j, \\ \beta_i - 1, & k = i, \\ \beta_j + 1, & k = j, \end{cases}$$

then $\beta!/\alpha! = (\alpha_i + 1)/\alpha_j = \beta_i/\alpha_j$ and

$$\mathbf{A}\varphi_\alpha = \frac{\alpha_j}{\sqrt{\alpha!}}\, x^\beta = \alpha_j \frac{\sqrt{\beta!}}{\sqrt{\alpha!}}\, \varphi_\beta = \alpha_j \frac{\sqrt{\beta_i}}{\sqrt{\alpha_j}}\, \varphi_\beta = \sqrt{\alpha_j \beta_i}\, \varphi_\beta.$$

Reciprocally, $\mathbf{A}^*\varphi_\beta = \beta_i\, x^\alpha/\sqrt{\beta!} = \cdots = \sqrt{\beta_i \alpha_j}\, \varphi_\alpha$. But since the vectors $\varphi_\alpha$ form an orthonormal basis,

$$(\mathbf{A}\varphi_\alpha, \varphi_\beta) = (\varphi_\alpha, \mathbf{A}^*\varphi_\beta) = \sqrt{\beta_i \alpha_j} \in \mathbb{R}$$

and all other matrix entries in the basis $\{\varphi_\alpha\}$ are zeros. Therefore the derivations $\mathbf{A}$ and $\mathbf{A}^*$ are mutually adjoint on $\mathcal{H}_m$.

2. Using the structure of the tensor product $\mathcal{D}_m = \mathcal{H}_m \otimes \mathbb{C}^n$, one can represent the commutators as follows:

$$\operatorname{ad}_A = \mathbf{A} \otimes E - \operatorname{id} \otimes A.$$

Indeed, for any element $\varphi v$, where $\varphi \in \mathcal{H}_m$ is a polynomial and $v \in \mathbb{C}^n$ a vector considered as a constant vector field on $\mathbb{C}^n$, by the Leibnitz rule

$$[\mathbf{A}, \varphi v] = (\mathbf{A}\varphi)v + \varphi[\mathbf{A}, v] = (\mathbf{A}\varphi)v - \varphi\, Av.$$

Obviously, because of the choice of the Hermitian structure on $\mathcal{H}_m \otimes \mathbb{C}^n$, the operator $\operatorname{id} \otimes A$ is adjoint to $\operatorname{id} \otimes A^*$ whereas the adjoint to $\mathbf{A} \otimes E$ is the tensor product of the adjoint to $\mathbf{A}$ by the identity. By the first statement of the lemma, the former is equal to $\mathbf{A}^*$, so that the adjoint to $[\mathbf{A}, \cdot]$ is $\mathbf{A}^* \otimes E - \operatorname{id} \otimes A^*$ which coincides with $[\mathbf{A}^*, \cdot] = \operatorname{ad}_{A^*}$. □

**Theorem 4.12** (G. Belitskii [**Bel79**]; see also [**Dum93, Van89**])**.** *A formal vector field $F(x) = Ax + V_2(x) + \cdots$ with the linearization matrix $A$ is formally equivalent to a vector field $F'(x) = Ax + V_2'(x) + \cdots$ whose nonlinear part commutes with the linear vector field $\mathbf{A}^*(x) = A^*x$:*

$$[F' - \mathbf{A}, \mathbf{A}^*] = 0. \tag{4.8}$$

*If the vector field $F$ is real (i.e., has only real Taylor coefficients, in particular, $A$ is real), then both the formal normal form and the conjugating transformation can be chosen real.*

**Proof.** The proof is based on the following well-known observation: if $L$ is a linear endomorphism of a complex Hermitian or real Euclidean space $H$ into itself, then the image of $L$ and the kernel of its Hermitian (resp., Euclidean) adjoint $L^*$ are orthogonal complements to each other:

$$(\operatorname{img} L)^\perp = \ker L^*.$$

It follows then that $\ker L^*$ is complementary to $\operatorname{img} L$ in $H$.

Indeed, $\xi \in (\operatorname{img} L)^\perp$ if and only if $(\xi, Lv) = 0$ for all $v \in H$, which means that any vector $v$ is orthogonal to $L^*\xi$. This is possible if and only if $L^*\xi = 0$.

Applying this observation to the operator $L_m = \operatorname{ad}_A$ restricted on any space $\mathcal{D}_m$ and using Lemma 4.11, we see that the subspaces $\mathcal{N}_m = \ker \operatorname{ad}_{A^*}|_{\mathcal{D}_m}$ are orthogonal (hence complementary) to the image of $L_m$ and therefore satisfy the assumption (4.5) of Theorem 4.7. Therefore all *nonlinear* terms $V_2, V_3, \dots$ can be chosen to commute with $\mathbf{A}^*(x) = A^*x$, which is in turn possible if and only if their formal sum, equal to $F - \mathbf{A}$, commutes with $\mathbf{A}^*$.

In the real case one has to replace the Hermitian spaces $\mathcal{H}_m$, $\mathbb{C}^n$ and $\mathcal{D}_m = \mathcal{H}_m \otimes_{\mathbb{C}} \mathbb{C}^n$ by their real (Euclidean) counterparts ${}^{\mathbb{R}}H_m$, $\mathbb{R}^n$ and ${}^{\mathbb{R}}\mathcal{D}_m = {}^{\mathbb{R}}\mathcal{H}_m \otimes_{\mathbb{R}} \mathbb{R}^n$. Then for any real matrix $A$ the image of the commutator $\operatorname{ad}_A$ and the kernel of $\operatorname{ad}_{A^*}$, where $A^*$ is a transposed matrix, are orthogonal and hence complementary. Then the homological equation $\operatorname{ad}_A P_m = V_m' - V_m$ can be solved with respect to $P_m \in {}^{\mathbb{R}}\mathcal{D}_m$ and $V_m' \in \ker \operatorname{ad}_{A^*} \cap {}^{\mathbb{R}}\mathcal{D}_m$ when $V_m \in {}^{\mathbb{R}}\mathcal{D}_m$. The Poincaré–Dulac paradigm does the rest of the proof. □

This general statement immediately implies a number of corollaries.

**Example 4.13.** If $A$ is a diagonal matrix with the spectrum $\{\lambda_1, \dots, \lambda_n\}$, then $A^*$ is also diagonal with the conjugate eigenvalues $\{\bar\lambda_1, \dots, \bar\lambda_n\}$. As was already noted, restriction of $\operatorname{ad}_{A^*}$ on $\mathcal{D}_m$ is diagonal with the eigenvalues $\langle\bar\lambda, \alpha\rangle - \bar\lambda_k = \overline{\langle\lambda, \alpha\rangle - \lambda_k}$. Its kernel consists of the same resonant monomials as defined previously, so in this case Theorem 4.12 yields the usual Poincaré–Dulac form.

Sometimes, diagonalization of the linear part is nonconvenient (especially for real vector fields). In such a case Theorem 4.12 may yield a simple real normal form.

**Example 4.14.** If $I = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right) = -I^*$ is the matrix of rotation on the *real* plane $\mathbb{R}^2$ with the coordinates $(x, y)$, then $\ker \operatorname{ad}_{I^*} = \ker \operatorname{ad}_I$ and the *entire* formal normal form, including the linear part, commutes with the rotation vector field $\mathbf{I} = x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}$. Any such rotationally symmetric real vector field must necessarily be of the form

$$f(x^2+y^2)\bigl(x\tfrac{\partial}{\partial x} + y\tfrac{\partial}{\partial y}\bigr) + g(x^2+y^2)\bigl(x\tfrac{\partial}{\partial y} - y\tfrac{\partial}{\partial x}\bigr), \tag{4.9}$$

where $f(r), g(r) \in \mathbb{R}[[r]]$ are two real formal series in one variable. Indeed, $A$ commutes with itself and the radial (Euler) vector field $\mathbf{E} = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}$ which form a basis at all nonsingular points; a linear combination $f\mathbf{E} + g\mathbf{I}$ with $f, g$ scalar coefficients, commutes with $\mathbf{I}$ if and only if $\mathbf{I}f = \mathbf{I}g = 0$, that is, if $f$ and $g$ are constants on all circles $x^2 + y^2 = r^2$.

The linear part is of the prescribed form if $f(0) = 0$, $g(0) = 1$. Since $g$ is formally invertible, the normal form (4.9) is formally orbitally equivalent to the formal vector field

$$\begin{aligned} F' &= \mathbf{I} + f(x^2+y^2)\mathbf{E}, \qquad f \in \mathbb{R}[[u]], \quad f(0) = 0, \\ \mathbf{I} &= x\tfrac{\partial}{\partial y} - y\tfrac{\partial}{\partial x}, \qquad \mathbf{E} = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}, \end{aligned} \tag{4.10}$$

with a formal series $f(u)$ in the resonant monomial $u = x^2 + y^2$.

Note that the "standard" demonstration of this result via preliminary diagonalization of $A$ requires that all subsequent Poincaré–Dulac transformations be preserving the complex conjugacy, which is an additional independent condition.

The same observation explains why the normal form is so often explicitly integrable.

**Corollary 4.15.** *Assume that the matrix $A \neq 0$ is* normal, *i.e., it commutes with the adjoint matrix $A^*$. Then the vector field can be formally transformed to a field which commutes with the (nontrivial) linear vector field $\mathbf{A}^*$.* □

Indeed, in this case from (4.8) and $[\mathbf{A}, \mathbf{A}^*] = 0$ it follows that $[F, \mathbf{A}^*] = 0$. This observation allows us to decrease the dimension of the system; cf. with §4**J**.

**Remark 4.16.** We wish to stress that *there is no distinguished Hermitian structure on* $\mathbb{C}^n$. One can choose this structure arbitrarily and only then the standard Hermitian structure appears on $\mathcal{H}_m$ and $\mathcal{D}_m$. Thus the assumption of this corollary is not restrictive, in particular, it always holds whenever $A$ is diagonalizable.

**4F. Parametric case.** The Poincaré–Dulac method of normalization of any finite jet or the entire Taylor series, involves only the *polynomial* (*ring*) operations (additions, subtractions and multiplications) with the Taylor coefficients of the original field, *except for inversion of the operator* $\operatorname{ad}_A$. This allows us to construct formal normal forms depending on parameters.

**Definition 4.17.** A formal series $f \in \mathbb{C}[[x]]$ is said to depend *polynomially* on finitely many parameters $\lambda = (\lambda_1, \dots, \lambda_m) \in \mathbb{C}^n$, if each coefficient depends polynomially on $\lambda$,

$$f = \sum_\alpha c_\alpha x^\alpha, \qquad c_\alpha \in \mathbb{C}[\lambda].$$

No assumption on the degrees $\deg c_\alpha$ is made.

The formal series $f = \sum c_\alpha x^\alpha \in \mathbb{C}[[x]]$ depends (strongly) *analytically* on the parameters *in a domain* $\lambda \in U$, if each coefficient $c_\alpha$ of this series depends on the parameters analytically in the *common domain* $U \subseteq \mathbb{C}^m$, $c_\alpha \in \mathcal{O}(U)$.

The formal series $f = \sum c_\alpha x^\alpha$ *weakly analytically* depends on the parameters $\lambda \in (\mathbb{C}^m, 0)$, if each coefficient $c_\alpha$ is a germ of analytic function, $c_\alpha \in \mathcal{O}(\mathbb{C}^m, 0)$. In contrast to the previously defined analytic dependence, intersection of all domains where representatives of the germs $c_\alpha$ are defined, can reduce to the single point $\lambda = 0$.

We will use the common name *semiformal series* to denote elements from the algebras $\mathfrak{A}[[x]]$ in the above three cases when $\mathfrak{A} = \mathbb{C}[\lambda]$, $\mathfrak{A} = \mathcal{O}(U)$ and $\mathfrak{A} = \mathcal{O}(\mathbb{C}^n, 0)$ respectively.

**Theorem 4.18** (Formal normal form with parameters)**.**

1. *If the vector field* (*holomorphic or formal*) $F = F(\cdot, \lambda) = \mathbf{A}(\lambda) + F_2(\lambda) + \cdots$ *depends weakly analytically on parameters* $\lambda \in (\mathbb{C}^m, 0)$*, then by a formal transformation one can bring the field to the formal normal form* $F'$ *satisfying the condition*

$$[F' - \mathbf{A}, \mathbf{A}^*(0)] = 0, \tag{4.11}$$

*where* $\mathbf{A}(0)$ *is the linear vector field corresponding to* $\lambda = 0$*, and* $\mathbf{A}^*(0)$ *its adjoint linear field. Both the formal normal form* $F'$ *and the transformation* $H$ *reducing* $F$ *to* $F'$ *can be chosen weakly analytically depending on the parameters* $\lambda \in (\mathbb{C}^m, 0)$ *in the sense of Definition 4.17. If* $F$ *was real, then also* $F'$ *and* $H$ *can be chosen real.*

2. *If the linear part* $\mathbf{A}(\lambda) \equiv \mathbf{A}(0) \equiv \mathbf{A}$ *is constant* (*does not depend on* $\lambda$) *and the field itself depends polynomially or strongly analytically on the parameters* $\lambda \in U$*, then both the normal form* (4.11) *and the corresponding normalizing transformation can be chosen polynomially (resp., strongly analytically) depending on the parameters in the same domain.*

**Proof.** We start with a very general observation, basically, a geometrical reformulation of the Implicit Function theorem.

If $L\colon X \to Y$ is a linear map between vector spaces, which is *transversal* to a subspace $Z \subseteq Y$, then for any analytic or polynomial map $y\colon \lambda \mapsto y(\lambda)$, $\lambda \in U$ or $\lambda \in \mathbb{C}^n$, one can find two maps $x\colon \lambda \mapsto x(\lambda) \in X$ and $z\colon \lambda \mapsto z(\lambda) \in Z$, such that $Lx(\lambda) + z(\lambda) = y(\lambda)$. If in addition $L$ also depends on $\lambda$ and is transversal to $Z$ for $\lambda = 0$, then the solutions still can be found, but only locally for the parameter values $\lambda \in (\mathbb{C}^m, 0)$ sufficiently close to the origin. In this case analyticity of $x(\lambda), z(\lambda)$ in the larger domain $U$ or polynomiality in general may fail.

This observation can be applied to the homological operator $L = \operatorname{ad}_A$ acting in the space $X = \mathcal{D}_m$, and the subspace $Y = \mathcal{N}_m$ of homogeneous vector fields commuting with $\mathbf{A}^*(0)$. Holomorphic (polynomial) solvability of the homological equation on each step guarantees the possibility of transforming the field to the normal form with the required properties. □

**Remark 4.19** (Warning)**.** The difference between constant and nonconstant linearization matrices is rather essential in what concerns the size of the common domain of analyticity of all Taylor coefficients of the normal form and/or conjugating transformation.

Suppose that all coefficients of the analytic family $F(\lambda)$ of formal vector fields are defined and holomorphic in some *common* domain $U$ (e.g., the field is analytic in $D \times U$, where $D$ is a small polydisk).

If the linearization matrix of $F(\lambda)$ does not depend on the parameters, then by the second assertion of Theorem 4.18, one may remove from the expansion of $F$ all terms that are nonresonant (i.e., the terms that do not commute with the linear field $\mathbf{A}^*$ which is independent of the parameters). All coefficients of all series (the normal form and the conjugacy) will be holomorphic in the maximal natural domain $U$.

All the way around, if the linearized field $\mathbf{A}(\lambda)$ depends on parameters, then by a formal transformation one can eliminate all terms that are resonant with respect to $\mathbf{A}(0)$. The coefficients of the normal form and the transformation will still be analytically dependent on $\lambda$, but their domains should be expected to shrink as the degree of the corresponding terms grow.

Indeed, assume that the linear field $\mathbf{A}(0)$ is nonresonant. Then the formal normal form guaranteed by the first assertion of Theorem 4.18 is *linear*, $F' = \mathbf{A}(\lambda)$. Yet clearly for some values of the parameter $\lambda$ which are arbitrarily close to $\lambda = 0$, the spectrum of the matrix $A(\lambda)$ can become resonant, hence it will be impossible to eliminate completely all terms of the corresponding order. The apparent contradiction is easily explained: the domain of analyticity of the coefficient of a high order cannot be so large as to include values of the parameter corresponding to resonances of that order. Note that if $A(0)$ is nonresonant, then the possible order of resonances occurring for $A(\lambda)$ necessarily grows to infinity as $\lambda \to 0$.

**4G. Formal classification of self-maps.** Besides formal vector fields, formal isomorphisms act also on themselves by conjugacy: if

$$G(x) = Mx + V_2(x) + \cdots \in \mathrm{Diff}[[\mathbb{C}^n, 0]], \qquad \det M \neq 0, \tag{4.12}$$

is a formal self-map, then another formal self-map $H \in \mathrm{Diff}[[\mathbb{C}^n, 0]]$ transforms $G$ to $G' = H \circ G \circ H^{-1}$. In the same way as before, one may ask if all nonlinear terms $V_2, V_3, \dots$ can be removed from the expansion by applying a suitable formal conjugacy.

The strategy is the same as described in §4**B**. The polynomial transformation $H(x) = x + P_m(x)$ with a vector homogeneous nonlinearity $P_m$ of degree $m$ conjugates $G(x)$ as in (4.12) with a self-map $G'(x) = G(x) + R_m(x) + \cdots$, in which $R_m$ is a homogeneous vector polynomial of order $m$, implicitly defined by the identity

$$G(x) + P_m(G(x)) = G(x + P_m(x)) + R_m(x + P_m(x)) + \cdots . \tag{4.13}$$

After collection of terms of order $m$ this yields the equation

$$P(Mx) - MP(x) = R(x), \qquad P = P_m,\ R = R_m, \tag{4.14}$$

which we can attempt to solve with respect to $P$. This is the multiplicative analog of the homological equation (4.2). The operator

$$S_M \colon \mathcal{D}_m \to \mathcal{D}_m, \qquad P(x) \mapsto MP(x) - P(Mx), \tag{4.15}$$

can be studied by methods similar to the operator $\mathrm{ad}_A$. If $M$ is a diagonal matrix with the diagonal entries $\mu_1, \dots, \mu_n$, then all monomials $F_{k\alpha}$ of the standard basis in $\mathcal{D}_m$ are eigenvectors for $S_M$ with the eigenvalues $\mu_j - \mu^\alpha = \mu_j - \mu_1^{\alpha_1} \cdots \mu_n^{\alpha_n}$. If all these expressions are nonzero, the operators $S_M$ will always be invertible and hence the formal self-map $G$ will be formally linearizable. If some of the expressions $\mu_j - \mu^\alpha$ are zeros, then one can transform $G$ to a nonlinear normal form. All these results can be obtained in exactly the same way as for the formal vector fields.

**Definition 4.20.** A *multiplicative* resonance between nonzero complex numbers $\mu = (\mu_1, \dots, \mu_n) \in (\mathbb{C}^*)^n$ is an identity of the form

$$\mu_j - \mu^\alpha = 0, \qquad |\alpha| \geqslant 2,\ j = 1, \dots, n. \tag{4.16}$$

A nondegenerate matrix $M \in \mathrm{GL}(n, \mathbb{C})$ and a formal self-map $G(x) = Mx + \cdots \in \mathrm{Diff}[[\mathbb{C}^n, 0]]$ are nonresonant if there are no multiplicative resonances between the eigenvalues of $M$. A *multiplicative resonant monomial* corresponding to the resonance (4.16), is the vector whose $j$th component is $x^\alpha$ and all others are zeros.

**Theorem 4.21** (Poincaré–Dulac theorem for self-maps)**.** *Any invertible formal self-map is formally equivalent to a formal self-map whose linear part is in the Jordan normal form, and the nonlinear part contains only resonant monomials with complex coefficients. In particular, a nonresonant formal self-map is formally conjugated to the linear map $G'(x) = Mx$.* □

Rather obviously, Theorem 4.21 can be further elaborated and an analog of Belitskii Theorem 4.12 established. However, we will not deal with these matters and concentrate from now on on vector fields and automorphisms in low dimension (2 for fields, 1 for self-maps) which will be the principal tool in the rest of the book.

* * *

**4H. Cuspidal points.** One important case where Theorem 4.12 is considerably stronger than the Poincaré–Dulac Theorem 4.10 is that of vector fields with *nilpotent* linear parts, which are "maximally nondiagonalizable". In this case *all* monomials will be resonant and Theorem 4.10 is void. We will only consider the planar case where the linear part is the vector field $J = y\frac{\partial}{\partial x} \in \operatorname{Mat}(2,\mathbb{R})$ (the linearization matrix is a nilpotent Jordan cell of size 2). From Theorem 4.12 we can immediately derive the following corollary.

**Theorem 4.22.** *A vector field on the plane with the linear part $J = y\frac{\partial}{\partial x}$ is formally equivalent to the vector field*

$$J + b(x)E + a(x)\tfrac{\partial}{\partial y}, \qquad a, b \in \mathbb{C}[[x]],\ E = x\tfrac{\partial}{\partial x} + y\tfrac{\partial}{\partial y}, \tag{4.17}$$

*with the formal series $a, b \in \mathbb{C}[[x]]$ in one variable $x$ starting with terms of order* 2 *and* 1 *respectively.*

**Proof.** We need only to describe the kernel of the operator $\operatorname{ad}_{J^*}$, where $J^* = x\frac{\partial}{\partial y}$ is the "adjoint" vector field. The kernel of the operator $\operatorname{ad}_{J^*} = [x\frac{\partial}{\partial y}, \cdot\,]$ restricted on $\mathcal{D}_m$ can be immediately computed. Indeed,

$$[x\tfrac{\partial}{\partial y}, u\tfrac{\partial}{\partial x} + v\tfrac{\partial}{\partial y}] = xu_y\tfrac{\partial}{\partial x} + (xv_y - u)\tfrac{\partial}{\partial y},$$

and the commutator vanishes only if both $u$ and hence $v_y$ depend only on $x$. Since both $u, v$ must be homogeneous of degree $m$, we conclude that

$$\ker \operatorname{ad}_{J^*}\big|_{\mathcal{D}_m} = \beta(x^m\tfrac{\partial}{\partial x} + x^{m-1}y\tfrac{\partial}{\partial y}) + \alpha x^m\tfrac{\partial}{\partial y} = \beta x^m\,(x\tfrac{\partial}{\partial x} + y\tfrac{\partial}{\partial y}) + \alpha x^m\tfrac{\partial}{\partial y}$$

for some constants $\alpha = \alpha_m$ and $\beta = \beta_m$ which will be the coefficients of the respective series $a, b$. □

Yet the complementary subspaces $\mathcal{N}_m$ may be chosen in a different way, not necessary as prescribed by Theorem 4.12. This may be more convenient for some applications.

**Theorem 4.23.** *The planar formal vector field with the linear part* $J = y\frac{\partial}{\partial x}$, *is formally equivalent to the vector field*

$$J + [yb(x) + a(x)]\tfrac{\partial}{\partial y}, \tag{4.18}$$

*where* $a(x)$ *and* $b(x)$ *are two formal series of orders* 2 *and* 1 *respectively.*

**Proof.** We reduce this assertion directly to the general Poincaré–Dulac paradigm. The image of $\operatorname{ad}_J$ in $\mathcal{D}_m$ can be complemented by the 2-dimensional space $\mathcal{N}'_m$ of vector fields $(\alpha x^m + \beta x^{m-1}y)\frac{\partial}{\partial x}$, as noted in [**Arn83**, §35 D]. Indeed, the condition $[y\frac{\partial}{\partial x}, f\frac{\partial}{\partial x} + g\frac{\partial}{\partial y}] = u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y}$ takes the form of the system of linear partial differential equations

$$yf_x - g = u, \qquad yg_x = v.$$

While it can be not solvable for some $u, v$, the system of equations

$$yf_x - g = u, \quad yg_x + \alpha x^m + \beta x^{m-1}y = v \tag{4.19}$$

can be always resolved for any pair of homogeneous polynomials $u, v \in \mathbb{C}[x, y]$ of degree $m$ and the constants $\alpha, \beta$. To see this, apply $y\frac{\partial}{\partial x}$ to the first equation:

$$y^2 f_{xx} = yu_x + v - \alpha x^m - \beta x^{m-1}y.$$

The equation $y^2 f_{xx} = w$ is uniquely solvable for any monomial $w$ divisible by $y^2$. On the other hand, the constants $\alpha, \beta$ can be found to guarantee that the terms proportional to $x^m$ and $x^{m-1}y$ in the right hand side of this equation vanish. This choice automatically guarantees solvability of the second equation in (4.19) as well. The constants found in this way, appear as coefficients of the respective series $a, b$. □

**4I. Vector fields with zero linear parts.** If the formal vector field $F$ starts with $k$th order terms, $F(x) = V_k(x) + V_{k+1}(x) + \cdots$, $k \geqslant 2$, then application of the formal transformation $H(x) = x + P_2(x)$ conjugates $F$ with the vector field $F'(x) = V_k + V'_{k+1} + \cdots$ with the same (nonlinear) principal part $V_k$, if

$$V_k(x) + V_{k+1}(x) + \left(\frac{\partial P_2}{\partial x}\right) V_k(x) + \cdots = V_k(x + P_2(x)) + V'_{k+1}(x + P_2(x)) + \cdots$$

which after collecting the homogeneous terms of order $k + 1$ yields

$$[V_k, P_2] = V_{k+1} - V'_{k+1}.$$

If this equation is resolved for a suitably chosen $V'_{k+1}$ (e.g., equal to zero if that is possible), one can pass to terms of order $k + 2$ by applying a transform of the form $H(x) = x + P_3(x)$ which does not affect the terms of order $V_k$ and $V_{k+1}$ and so on. As a result, one has to resolve in each order the homological equation

$$\operatorname{ad}_{V_k} P_m = V_{m+k-1} - V'_{m+k-1} \tag{4.20}$$

with respect to the homogeneous vector field $P_m$ of degree $m$. As before, complete elimination of all nonprincipal terms of orders $k+1$ and more, is possible if the operator $\operatorname{ad}_{V_k}$ is surjective, otherwise it will be necessary to introduce the "normal subspaces" $\mathcal{N}_{m+k-1} \subset \mathcal{D}_{m+k-1}$ complementary to the image $\operatorname{ad}_{V_k}(\mathcal{D}_m) \subseteq \mathcal{D}_{m+k-1}$ and choose the components $V'_{m+k-1}$ of the formal normal form from these subspaces.

In contrast to the case $k = 1$ discussed earlier, the operator $\operatorname{ad}_{V_k}$ *increases the degrees*, i.e., acts between *different* spaces, the dimension of the target space in general being higher than that of the source space. Thus the number of parameters in the normal form will be infinite. A notable exception is the one-dimensional case $\dim x = 1$.

**Theorem 4.24.** *A nonzero formal vector field from* $\mathcal{D}[[\mathbb{C}, 0]]$ *is formally equivalent to one of the vector fields of the form*

$$\bigl(x^{k+1} + ax^{2k+1}\bigr)\tfrac{\partial}{\partial x}, \qquad k \in \mathbb{N},\ a \in \mathbb{C}. \tag{4.21}$$

**Proof.** Any nonzero formal vector field on $\mathbb{C}^1$ starts with the term $a_{k+1}x^{k+1}\frac{\partial}{\partial x}$, $a_{k+1} \neq 0$. One can make $a_{k+1}$ equal to 1 by a linear transformation $x \mapsto cx$, if the ground field is $\mathbb{C}$.

In this case all spaces $\mathcal{D}_m$ are one-dimensional, and the commutator with the principal term $x^{k+1}\frac{\partial}{\partial x}$ can be immediately computed:

$$\bigl[x^{k+1}\tfrac{\partial}{\partial x}, x^m\tfrac{\partial}{\partial x}\bigr] = (k-m+1)x^{k+m}\tfrac{\partial}{\partial x}. \tag{4.22}$$

This operator is surjective for all $m \neq k+1$. Thus only the term $x^{2k+1}\frac{\partial}{\partial x}$ cannot be eliminated. $\square$

Note that over the field of reals $\mathbb{R}$ the normal form is different: if $k$ is *even*, then by the *real* homothety one can make the principal coefficient only $\pm 1$,

$$\bigl(\pm x^{k+1} + ax^{2k+1}\bigr)\tfrac{\partial}{\partial x}, \qquad k \in \mathbb{N},\ a \in \mathbb{R}.$$

For odd $k$ the fields with different signs are equivalent (transformed into each other by the symmetry $x \mapsto -x$).

**Remark 4.25.** In fact, the above arguments show that *any two formal vector fields on the line having a zero of multiplicity $k+1$ at the origin and common $(2k+1)$-jet, are formally equivalent.*

It is sometimes more convenient instead of the polynomial normal form (4.21) to use the *rational* formal normal form

$$\frac{x^{k+1}}{1-ax^k}\cdot\frac{\partial}{\partial x},\qquad k\in\mathbb{N},\ a\in\mathbb{C}. \tag{4.23}$$

This (rational) field is *analytically* equivalent to the field (4.21) with the same $a$. On the other hand, two vector fields in the normal form (4.23) with different values of $a$ cannot be equivalent, as will be shown in §6$\mathbf{B}_2$.

**Theorem 4.26.** *Any self-map* $x \mapsto x + x^{k+1} + \dots$, $k \in \mathbb{N}$, *tangent to identity, is formally equivalent to*:

(1) *the time one map of the polynomial vector field* (4.21),
(2) *the time one map of the rational vector field* (4.23),
(3) *the polynomial map* $x \mapsto x + x^{k+1} + ax^{2k+1}$,

*with the same complex parameter* $a \in \mathbb{C}$ *which is the formal invariant of the classification together with the order* $k+1$.

**Proof.** One can prove this result in exactly the same way as Theorem 4.24, namely, modifying the Poincaré–Dulac paradigm for the equation (4.13) and using the computation from Proposition 6.11 below.

Yet one can circumvent this parallel construction by reference to the formal embedding Theorem 3.17. Indeed, any formal self-maps from $\operatorname{Diff}[[\mathbb{C},0]]$ tangent to the identity with some order $k+1$ can be represented as a time one formal flow of a formal vector field from $\mathcal{D}[[\mathbb{C},0]]$. This field in turn can be brought to one of the two formal normal forms or to the formal (nonpolynomial!) field generating the polynomial normal form. □

**4J. Formal normal forms of elementary singular points on the real plane.** In this section we summarize the (orbital) formal normal forms for all planar (i.e., for $n = 2$) *real* vector fields with nonzero linear parts. Recall that two formal vector fields $F, F' \in \mathcal{D}[[\mathbb{R}^2,0]]$ are called *orbitally formally equivalent*, if there exist an invertible real formal series $\varphi \in \mathbb{R}[[x,y]]$, $\varphi(0,0) \neq 0$, such that $F$ is formally equivalent to $\varphi \cdot F'$, and the corresponding formal self-map has all real coefficients, i.e., belongs to the group $\operatorname{Diff}[[\mathbb{R}^2,0]]$. We use everywhere the term *singularity* to denote jets or germs of analytic vector fields or formal vector fields at the origin, depending on the context.

**Definition 4.27.** A singularity of the planar vector field is *elementary*, if at least one of the eigenvalues $\lambda_{1,2}$ of its linearization matrix is nonzero.

The only nonelementary singularity that has nonzero linearization matrix with both zero eigenvalues, is called *cuspidal*, or *nilpotent* singularity.

Real elementary points can be of several types that exhibit essentially different properties.

**Definition 4.28.** An elementary singularity is a *resonant node*, if the ratio of its eigenvalues is a natural or inverse natural number. The singularity is a *resonant saddle*, if both eigenvalues are real and their ratio is negative rational. A singularity is *elliptic*, if $\lambda_{1,2} = \pm i\omega$, $\omega > 0$. Finally, the singularity is a *saddle-node*, if exactly one eigenvalue is zero.

**Proposition 4.29** (Formal normal forms of planar singularities)**.** *By a real orbital formal transformation from the group* $\operatorname{Diff}[[\mathbb{R}^1, 0]] \times \operatorname{Diff}[[\mathbb{R}^2, 0]]$ *any real formal vector field* $\mathcal{D}[[\mathbb{R}^2, 0]]$ *appearing in Table I.1, can be brought to the normal form from the right column of this table.*

**Proof.** Most of these results are particular cases of the general results proved earlier for the ground field $\mathbb{C}$, modulo the following obvious remark. If the linear part of the vector field can be brought into its Jordan normal form by a *real* linear transformation, then all results of the formal classification remain valid if the ground field is replaced by $\mathbb{R}$. The only nontrivial case where a real matrix cannot be normalized over $\mathbb{R}$ is that of the *elliptic* singular points whose linear part is linear rotation $\omega x \frac{\partial}{\partial y} - \omega y \frac{\partial}{\partial x}$, with the eigenvalues $\pm i\omega$. From the complex point of view this is a resonant saddle, yet diagonalization of this matrix requires enlarging the ground field. The alternative treatment of the elliptic case is explained in Example 4.14.

The assertion concerning saddle-nodes is a combination of the Poincaré–Dulac theorem and Theorem 4.24. While the condition $\lambda_2 = 0$ is not a resonance, it implies infinitely many resonances $\lambda_j = \lambda_j + m$ for any $m \in \mathbb{N}$. By the Poincaré–Dulac theorem, the field is formally equivalent to the field $xf(y)\frac{\partial}{\partial x} + yg(y)\frac{\partial}{\partial y}$ with $f(0) \neq 0$ and $g(0) = 0$ (otherwise the singular point cannot be elementary degenerate). Dividing by the invertible series $f(y)$ one can assume that $f \equiv 1$ and the variables (formally) separate. It remains to make the formal change of the variable $y$ which puts the one-dimensional vector field $g(y)\frac{\partial}{\partial y}$ into the normal form (4.21).

The saddle case is analyzed similarly: the identity $\langle \lambda, m \rangle = 0$ itself is not a resonance, but its integer multiple can be added to the right hand side of each of the identities $\lambda_1 = \lambda_1$ or $\lambda_2 = \lambda_2$, thus producing infinitely many resonances. Without loss of generality we assume that $\lambda_1 = -p$, $\lambda_2 = q$, $p, q \in \mathbb{N}$. Clearly, there are no other resonances and the Poincaré–Dulac normal form looks like $-pxf(u)\frac{\partial}{\partial x} + qy\,g(u)\frac{\partial}{\partial y}$, $f(0) = g(0) = 1$, where $u = x^p y^q$ is the resonant monomial. Passing to an orbitally equivalent system, one can assume that $f \equiv 1$.

| **Type** | **Conditions** | **Formal normal form** |
|---|---|---|
| Nonresonant | $[\lambda_1 : \lambda_2] \notin \mathbb{Q}$ or $\lambda_1 = \lambda_2 \neq 0$ | Linear |
| Resonant node | $[\lambda_1 : \lambda_2] = [r : 1]$, $r \in \mathbb{N}$, $r \geqslant 2$ | $\dot{x} = rx + ay^r$, $\dot{y} = y$ <br> $a \in \mathbb{C}$ formal invariant. |
| Resonant saddle (orbital) | $[\lambda_1 : \lambda_2] = -[p : q]$, $p, q \in \mathbb{N}$, not formally orbitally linearizable | $\dot{x} = -px$, $\dot{y} = \quad qy(1 \pm u^r + au^{2r})$, $u = x^q y^p$, <br> $r \in \mathbb{N}$, $a \in \mathbb{R}$ formal orbital invariants |
| Elliptic points (orbital) | $\lambda_{1,2} = \pm i\omega$, not formally orbitally linearizable | $\dot{x} = \quad y \pm x(u^r + au^{2r})$, $\dot{y} = -x \pm y(u^r + au^{2r})$, <br> $u = x^2 + y^2$, $a \in \mathbb{R}$ formal orbital invariant |
| Saddle-node (orbital classification) | $\lambda_1 \neq 0$, $\lambda_2 = 0$, formally isolated singularity | $\dot{x} = x$, $\dot{y} = \pm y^{r+1} + ay^{2r+1}$, <br> $r \in \mathbb{N}$, $a \in \mathbb{R}$ formal orbital invariants |
| Cuspidal (nilpotent) point (nonelementary) | Nonvanishing linearization matrix with two zero eigenvalues | $\dot{x} = y$, $\dot{y} = a(x) + yb(x)$, <br> $a, b \in \mathbb{R}[[x]]$ two formal series, $\operatorname{ord} a \geqslant 2$, $\operatorname{ord} b \geqslant 1$. |
| One-dimensional degenerate vector field | $\lambda = 0$, formally isolated singularity | $\dot{x} = \pm x^{r+1} + ax^{2r+1}$, <br> or $\dot{x} = \pm \dfrac{x^{r+1}}{1 - ax^r}$, <br> $r \in \mathbb{N}$, $a \in \mathbb{C}$ formal invariants |

**Table I.1.** Formal normal forms for real vector fields. All rows of the table, except the last one, refer to planar formal vector fields and give orbital formal normal forms.

The field in the Poincaré–Dulac normal form admits the projection $\mathbb{R}^2 \to \mathbb{R}^1$, $(x, y) \mapsto u = x^p y^q \in \mathbb{R}^1$. The projected system has the form

$$\dot{u} = uF(u), \qquad F(u) = g(u) - 1, \tag{4.24}$$

called the *quotient equation*. By a suitable formal transformation $u \mapsto u' = u(1+h(u))$, $h(0)=0$, the quotient vector field can be brought to the form (4.21), corresponding to $g(u) = 1 + u^{k-1} + au^{2k-1}$. It remains to observe that any formal transformation of the variable $u \mapsto u[1+h(u)]$, $h(0)=0$, can be "covered" by the transformation $(x,y) \mapsto (x', y'(x,y))$, where

$$x' = x, \qquad y' = y[1 + h(x^p y^q)]^{1/q} \in \mathbb{R}[[x,y]],$$

re-expanding the invertible series in square brackets into the binomial series. This transformation brings the initial field into the required formal normal form.

The same construction almost literally applies to the elliptic case: the infinite formal normal form (4.10) admits projection onto the $u$-axis with $u = x^2 + y^2$, and the quotient equation takes the form $\dot u = 2uf(u)$. We leave it as an exercise to prove that any formal line transformation $u \mapsto u[1+h(u)]$, $h(0)=0$, can also be "covered" by a suitable *real* plane formal transformation. □

**Remark 4.30.** If necessary, the polynomial normal forms from Table I.1 can be replaced by rational normal forms involving the rational normal form for one-dimensional quotient vector fields.

Note also that all normal forms of *elementary* singularities from this table are integrable: the quotient equation can be explicitly integrated in quadratures (especially easily if it has the rational normal form (4.23)). After this integration the variables $x$ and $y$ always separate. This integrability will be repeatedly used in the rest of the book to produce explicit computations with normal forms.

The cuspidal normal form is the famous Liénard system, corresponding to one of the simplest nonlinear and nonintegrable vector fields for which questions on the number of *limit cycles* is highly nontrivial. The Liénard system is sometimes written under the form

$$\dot x = y - f(x), \qquad \dot y = -g(x),$$

or as a second order scalar differential equation.

**Remark 4.31.** The dynamic (full, nonorbital) formal normal form contains more parameters than indicated in Table I.1. For instance, for the saddle-node the formal normal form is

$$\begin{cases} \dot x = x(\lambda_1 + b_1 y + \cdots + b_k y^k), \\ \dot y = y^{k+1}k + ay^{2k+1}, \qquad \lambda_1, b_1, \dots, b_k, a \in \mathbb{C}. \end{cases} \tag{4.25}$$

To prove this formula, we reduce the vector field to the form $xf(y)\frac{\partial}{\partial x} + g(y)\frac{\partial}{\partial y}$ as above and then by a suitable change of the variable $y$ only put $g$ into the standard form $g(y) = y^{k+1} + ay^{2k+1}$. The function $f(x)$ can be further simplified by transformations of the form $(x,y) \mapsto (h(y)x, y)$, $h(0) \neq 0$, preserving the second component: one immediately verifies that such a transformation results in replacing the series $f = f(y) \in \mathbb{C}[[y]]$ by another series

$$f' = f + \frac{g}{y} \cdot \frac{dh}{dy} = f + (y^{k+1} + ay^{2k+1}) \frac{d}{dy} \ln h.$$

Since $g$ begins with terms of order $k+1$, the difference between $f$ and $f'$ is necessarily $k$-flat (the logarithmic derivative $\frac{d}{dy} \ln h$ in the above formula is a well defined formal series from $\mathbb{C}[[y]]$ since $h(0)$ is nonvanishing). On the other hand, if the difference $f - f'$ is divisible by $y^{k+1}$, the quotient can be represented as the logarithmic derivative of a suitable series $h \in \mathbb{C}[[y]]$. Thus all terms of order $k+1$ and above can be eliminated from $f$ by the formal transformation.

A similar result can be formulated for resonant saddles and elliptic singularities.

## Exercises and Problems for §4.

A complex tuple $\lambda = (\lambda_1, \dots, \lambda_n) \in \mathbb{C}^n$ is called single-resonant, if all resonances between the components of this tuple follow from a single integer identity

$$\langle \alpha, \lambda \rangle = 0, \qquad \alpha \in \mathbb{Z}_+^n, \ \alpha \neq 0. \tag{4.26}$$

**Problem 4.1.** Describe the formal normal form of a vector field with a single-resonant spectrum of the linearization matrix. Show that this normal form is integrable in quadratures.

**Problem 4.2.** Describe all linear maps that preserve the formal normal form in Problem 4.1.

**Problem 4.3.** Describe the real formal normal forms for vector fields in $\mathbb{R}^3$ with the spectrum $0, \pm i\omega$.

**Problem 4.4.** The same question for fields in $\mathbb{R}^4$ with the spectrum $\pm i\omega_1$, $\pm i\omega_2$, if the ratio $\omega_1/\omega_2$ is irrational.

**Problem 4.5.** Describe symmetries of the formal normal forms in the Problems 4.3 and 4.4.

**Exercise 4.6.** Prove that if $F$ is a resonant formal vector field, then $\exp tF$ is a multiplicative resonant formal self-map for any $t \neq 0$. Is the inverse true?

**Problem 4.7.** Construct a formal normal form for vector fields in $\mathbb{C}^3$ with the nilpotent Jordan linear part $J = y\frac{\partial}{\partial x} + z\frac{\partial}{\partial y}$.

*Answer*: $J + a(x,u)E + b(x,u)F + c(x,u)F'$, where $E = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z}$ is the Euler field in three dimensions, $F = x\frac{\partial}{\partial y} + y\frac{\partial}{\partial z}$, $F' = \frac{\partial}{\partial z}$, and $u = u(x,y,z) = 2xz - y^2$.

**Exercise 4.8.** Find a formal normal form for a saddle-nodal self-map with the spectrum $(1, \mu)$, $|\mu| \neq 1$, in two dimensions.

**Problem 4.9.** Give a complete proof of the Poincaré–Dulac theorem for self-maps (Theorem 4.21).

**Problem 4.10.** Prove that the formal normal form of any vector field in the Poincaré domain is integrable in quadratures.