First aid

A. Crash course on functions of several complex variables

In this appendix we collect several facts about holomorphic functions of several variables. They can be found in a number of sources, among which we recommend the books [**Hör00**], [**GH78**], [**GR65**], [**Sha92**], [**Chi89**], and more recently the textbooks [**FG02**] and especially [**Ebe07**].

A.1. Holomorphic functions of several variables. A complex function $f(z_1,\dots,z_n)$ defined on an open domain U of the complex n-space $\mathbb{C}^n$ is *holomorphic* or *analytic* (these words will be used as complete synonyms) in U, if the real and imaginary parts of the function are differentiable at every point $a \in U$, and the differential $df_a \colon \mathbf{T}_a U \to \mathbb{C}$, is $\mathbb{C}$-linear:

$$df_a(\lambda\xi) = \lambda \cdot df_a(\xi) \qquad \forall \xi \in \mathbf{T}_a U \cong \mathbb{C}^n. \tag{A.1}$$

This condition can be written in the form of a system of partial differential equations called the Cauchy–Riemann equations,

$$\frac{\partial}{\partial \bar z_j} f = 0, \quad j = 1,\dots,n, \qquad \frac{\partial}{\partial \bar z_j} = \frac{1}{2}\left(\frac{\partial}{\partial x_j} - \frac{1}{i}\frac{\partial}{\partial y_j}\right). \tag{A.2}$$

Functions holomorphic in the domain U form a linear space which will be denoted by $\mathcal{O}(U)$.

We will often use the space $\mathcal{A}(U)$ of functions holomorphic in U and continuous on the closure $\overline{U}$; cf. with §1**B**. This space is equipped with the norm

$$\mathcal{A}(U) = \mathcal{O}(U) \cap C(\overline{U}), \qquad \|f\|_U = \max_{z\in\overline{U}} |f(z)| \quad \forall f \in \mathcal{A}(U). \tag{A.3}$$

A function of several variables is holomorphic if and only if it is holomorphic in each variable separately (Hartogs theorem).

A.2. Holomorphic maps and their inversion. A map $f\colon U \to \mathbb{C}^m$ is holomorphic, if all its components are holomorphic. Differentials of holomorphic maps are $\mathbb{C}$-linear maps from $\mathbf{T}_a U \cong \mathbb{C}^n$ to $\mathbf{T}_{f(a)}\mathbb{C}^m \cong \mathbb{C}^m$ for all $a \in U$. Since the composition of $\mathbb{C}$-linear maps is again $\mathbb{C}$-linear, composition of holomorphic maps is again a holomorphic map.

If the differential df_a of a holomorphic map $f\colon U \to \mathbb{C}^n$, $U \subseteq \mathbb{C}^n$, is invertible (as a $\mathbb{C}$-linear map of $\mathbb{C}^n$ into itself), then the map is locally invertible: there exists a holomorphic map g, defined in some neighborhood of $f(a)$, such that $g \circ f = \mathrm{id}$.

For holomorphic maps the implicit function theorem holds. If $U \subset \mathbb{C}^{n+m}$ and $f\colon U \to \mathbb{C}^n$ is a holomorphic map, $f = f(z,w)$, such that the differential of f with respect to the first variable is invertible at some point $(a,b) \in U$, then the system of equations $f(z,w) = 0$ determines z as a holomorphic (vector) function of w in some neighborhood of $b \in \mathbb{C}^m$, such that $f(z(w),w) \equiv 0$. The condition of invertibility means that the matrix of partial derivatives $\bigl(\partial f_i/\partial z_j\bigr)_{i,j=1}^n$, has nonzero determinant at $(a,b) \in \mathbb{C}^{n+m}$.

A.3. Cauchy formula and its consequences. Let $D_r = D_r(a)$ be a polydisk of *polyradius* $r = (r_1,\dots,r_n)$, $r_j > 0$,

$$D_r(a) = \{(z_1,\dots,z_n) \in \mathbb{C}^n \colon |z_j - a_j| < r_j,\ j = 1,\dots,n\},$$

and D_r° its *skeleton*, the Cartesian product of the boundary circles,

$$D_r^\circ(a) = \{(z_1,\dots,z_n) \in \mathbb{C}^n \colon |z_j - a_j| = r_j,\ j = 1,\dots,n\}.$$

Note that the skeleton forms only a small fraction of the boundary ∂D_r.

Similarly to functions of one complex variable, a function holomorphic in a polydisk D_r as above and continuous on its closure, can be obtained from its values on the skeleton of the polydisk by the Cauchy integral formula,

$$f(a) = \frac{1}{(2\pi i)^n} \int\!\cdots\!\int_{D_r^\circ(a)} \frac{dz_1 \wedge \cdots \wedge dz_n}{(z_1 - a_1)\cdots(z_n - a_n)} \tag{A.4}$$

(the integral can be understood as an iterated integral).

The Cauchy integral formula implies numerous corollaries, the most important among them the possibility of expanding a holomorphic function in a *converging* Taylor series.

We use the standard multi-index notation: for an integer vector $\alpha = (\alpha_1,\dots,\alpha_n) \in \mathbb{Z}_+^n$ we denote

$$\alpha! = \alpha_1!\cdots\alpha_n!, \quad z^\alpha = z_1^{\alpha_1}\cdots z_n^{\alpha_n}, \quad \frac{\partial^\alpha}{\partial z^\alpha} = \frac{\partial^{\alpha_1}}{\partial z^{\alpha_1}}\cdots\frac{\partial^{\alpha_n}}{\partial z^{\alpha_n}}.$$

In these notations the integral representation (A.4) implies the *Cauchy inequalities*

$$\left|\frac{\partial^\alpha}{\partial z^\alpha} f(a)\right| \leqslant \frac{\|f\|_{D_r(a)}}{\alpha!\, r^\alpha}, \qquad \forall \alpha \in \mathbb{Z}_+^n,$$

which in turn guarantee that the Taylor series for f converges on $D_r(a)$, and this convergence is uniform on any smaller polydisk centered at a,

$$\forall z \in D_r(a), \quad f(z) = \sum_{|\alpha|=0}^{\infty} c_\alpha (z-a)^\alpha, \qquad c_\alpha = \frac{1}{\alpha!} \cdot \frac{\partial^\alpha f}{\partial z^\alpha}(a).$$

A.4. Weierstrass compactness principle. Another consequence of the Cauchy inequalities is the *Weierstrass compactness principle.* It asserts that a sequence of holomorphic functions $\{f_k\}_{k=1}^\infty \subseteq \mathcal{O}(U)$ uniformly convergent on a bounded domain $U \subset \mathbb{C}^n$ (with compact closure), has a holomorphic limit. This principle implies that the space $\mathcal{A}(U) = \mathcal{O}(U) \cap C(\overline{U})$ introduced in (A.3), is a Banach (*complete* normed) space. This completeness plays a central role throughout the book.

A.5. Germs of analytic functions. Germs of analytic functions at a given point, say, at the origin $0 \in \mathbb{C}^n$, form a commutative algebra over $\mathbb{C}$, denoted by $\mathcal{O}(\mathbb{C}^n, 0)$. This algebra is *local*: its unique maximal ideal $\mathfrak{m} \subset \mathcal{O}(\mathbb{C}^n, 0)$ consists of germs vanishing at the origin. Usually we ignore the difference between germs and their representatives (defined in sufficiently small domains) both in argumentation and in notations.

The ring of germs $\mathcal{O}(\mathbb{C}^n, 0)$ is *Noetherian*: any ascending chain of ideals in this ring eventually stabilizes. This implies that any ideal in this ring has finite basis (Hilbert's theorem).

Any germ can be factored as a product of finitely many irreducible germs; the irreducible factors are defined uniquely modulo multiplication by units (elements from $\mathcal{O}(\mathbb{C}^n, 0) \smallsetminus \mathfrak{m}$). A germ is *square-free*, if all its irreducible factors are pairwise distinct (modulo units).

A.6. Analytic sets. A subset $X \subset \mathbb{C}^n$ is analytic if in a neighborhood of each point $a \in \mathbb{C}^n$ it can be represented as common zero locus of several functions analytic at a. By Hilbert's theorem, the number of such functions can always be assumed finite. Analytic sets are sometimes referred to as analytic varieties; they are always closed.

A set is an analytic submanifold of codimension $k \leqslant n$, if near each point $a \in X$ it is a common zero locus of k functions holomorphic at a with linearly independent (over $\mathbb{C}$) differentials.

Analytic sets have rather regular structure even in the case where they are not submanifolds of $\mathbb{C}^n$. In particular, every analytic variety can be *stratified*, i.e., represented as (locally) finite union of strata X_k of different dimensions, such that

(1) each stratum X_k is an analytic submanifold in $\mathbb{C}^n$ of certain dimension d_k, and
(2) the closure of each stratum consists of itself and several strata of lower dimensions.

One may in fact guarantee that the tangent planes to strata near the boundary points have certain limit positions compatible with that of tangent planes to the adjoining strata (Conditions A and B of Whitney). For most purposes one can use the characteristic property formulated in terms of transversality: any smooth map transversal to a stratum X_k at a point $a \in X_k$, is transversal also to all strata of higher dimensions which have a at their closure, at all points sufficiently close to a.

The principal stratum of highest dimension is called the *regular part* or set of *regular points* of X and denoted $\operatorname{Reg} X$.

The germ (X, a) of an analytic set X at a point $a \in \mathbb{C}^n$ is *irreducible*, if it cannot be represented as the union of two germs of analytic sets $X = X_1 \cup X_2$, such that $\operatorname{Reg} X_i \subsetneqq \operatorname{Reg} X$. The germ of a hypersurface $X = \{f = 0\}$ generated by an irreducible germ $f \in \mathcal{O}(\mathbb{C}^n, a)$, is irreducible. Regular parts of irreducible sets are locally connected.

Any germ of an analytic hypersurface admits an *irreducible decomposition* into the union of uniquely defined irreducible components of codimension 1. This follows from an irreducible factorization of holomorphic germs; see §**A.5**.

A.7. Uniformization. An analytic submanifold X of codimension k in $\mathbb{C}^n$ admits local uniformization near each point $a \in X$: there exists a holomorphic map $(\mathbb{C}^{n-k}, 0) \to (X, a)$ which is one-to-one.

Among singular analytic varieties, only *analytic curves*, varieties of complex dimension 1 admit *uniformization*. Any *irreducible* germ of an analytic curve $(X, a) \subset (\mathbb{C}^n, 0)$ can be parameterized by a holomorphic one-to-one map.

A.8. Forced analytic continuation: erasing of singularities. For some domains $U \subset \mathbb{C}^n$, any function holomorphic in U, can be extended as a function analytic in a larger domain. This phenomenon is peculiar for holomorphic functions in more than one variable.

If $U \subseteq \mathbb{C}^n$ is an open domain and $K \Subset U$ its compact subset, then any function analytic in $U \smallsetminus K$, extends on the whole of U. This means that *compact holes in the domain can be always erased* (Poincaré–Hartogs).

If (X, a) is the germ of an analytic variety of codimension 1 (hypersurface) and f is a *locally bounded* (i.e., bounded in some neighborhood of every point) function holomorphic in the complement to X, then f can be extended on X while remaining analytic. This can be proved by a straightforward application of the Cauchy integral formula exactly in the one-dimensional case, if X is a nonsingular hypersurface.

If X is the germ of analytic variety of codimension $\geqslant 2$, then the condition of local boundedness can be dropped: any function holomorphic in the complement to X, can be extended on X while remaining holomorphic. For instance, any isolated singular point of a holomorphic function on the plane $\mathbb{C}^2$ can be erased.

A.9. Meromorphic functions. The ring of holomorphic germs has no divisors of zero, hence admits extension to the field of fractions denoted by $\mathcal{M}(\mathbb{C}^n, a)$. A representative of a meromorphic germ $f = g/h$, $g, h \in \mathcal{O}(U)$, is a holomorphic function on the complement to the zero locus $\{h = 0\}$ of the denominator, which can be extended as a holomorphic map to the Riemann sphere $\mathbb{P} = \mathbb{C} \cup \{\infty\}$ on the complement to the *indeterminacy locus* $\{f = 0,\ g = 0\}$ (common zeros of the numerator and denominator).

A *meromorphic function* in a domain U is a collection of local representations $f_\alpha = g_\alpha/h_\alpha$ in charts of an open covering $\mathfrak{U} = \{U_\alpha\}$ of U, such that on the intersections $U_{\alpha\beta}$ the equalities $g_\alpha h_\beta - g_\beta h_\alpha = 0$ (this definition can be literally used for holomorphic manifolds). Under certain global assumptions on U, there exists a single global representation $f = g/h$ with holomorphic $g, h \in \mathcal{O}(U)$. Meromorphic functions form a field denoted by $\mathcal{M}(U)$.

A function meromorphic on $U \setminus Y$, a complement to an analytic variety Y of codimension $\geqslant 2$, can be extended as a meromorphic function on U (Lévi theorem).

A.10. Analyticity vs. algebraicity. An analytic subvariety of a complex projective space $\mathbb{P}^n$ is an algebraic variety (Chow theorem). A meromorphic function on $\mathbb{P}^n$ is *rational*, ratio of two homogeneous polynomials of the same degree in the homogeneous coordinates on $\mathbb{P}^n$.

B. Elements of the theory of Riemann surfaces.

B.1. Riemann surfaces and algebraic curves. A Riemann surface is a complex manifold of dimension one. The principal examples are the complex line $\mathbb{C}$ itself, open domains in $\mathbb{C}$, the Riemann sphere $\mathbb{P} = \mathbb{C} \cup \{\infty\}$, smooth affine and projective algebraic curves.

A map $f\colon C \to C'$ between two Riemann surfaces is holomorphic if it is locally defined by a holomorphic function $z' = f(z)$ for any local holomorphic charts z, z' on C, C' respectively.

The zero locus C of a bivariate polynomial $\{P(x,y) = 0\} \subset \mathbb{C}^2$ is called an *affine algebraic curve*. It may be nonsmooth, yet there always exists a Riemann surface $\widetilde{C}$ and a map $\varphi\colon \widetilde{C} \to C$ such that any smooth point $b \in C$ has a unique preimage $a \in \widetilde{C}$ and the germ $\varphi_a\colon (\widetilde{C}, a) \to (C, b)$ is biholomorphic. The curve $\widetilde{C}$ is called *normalization* of C.

Existence of normalization for any algebraic curve (normalization theorem) may be easily proved using the local uniformization theorem from §**A.7** and the irreducible decomposition theorem for curves from §**A.6**. For curves with normal crossings see Problem 25.1.

The closure of an affine algebraic curve in the projective plane is called the *projective* algebraic curve. Projective curves also admit normalization which is a *compact* Riemann surface.

Conversely, *any compact Riemann surface is algebraic*. There are many ways to formalize this statement. One of them is the following. For any abstract compact Riemann surface S there exists a projective algebraic curve $C \subset \mathbb{P}^2$ for which S is a normalization: $S = \widetilde{C}$.

B.2. Genus and degree of an algebraic curve. For an affine algebraic curve C, there exists a unique (modulo constant factor) polynomial of minimal degree whose locus is C, which is called the minimal polynomial of C and of the projective closure of C. The *degree* of an affine (projective) curve is the degree of this minimal polynomial.

The degree of a projective algebraic curve $C \in \mathbb{P}^2$ is equal to the number of intersections between this curve and a generic line $\ell \subset \mathbb{P}^2$.

The *genus* of a projective algebraic curve is the (topological) genus of its normalization considered as a smooth 2-dimensional surface. The genus of the affine algebraic curve is the genus of its projective closure.

If $f\colon C \to C'$ is a holomorphic map between two compact Riemann surfaces, them it defines a ramified covering of C' over the set of *critical values* of f. Near each critical value $a \in C$ the map f has the form $z \mapsto z^k = z'$ for suitable choices of local charts $z, z' \in (\mathbb{C}, 0)$ on C, C' and some natural number $k = k_a \geqslant 1$ (we set for convenience $k_a = 1$ for a regular point $a \in C$). If m is the number of sheets of this covering and g, g' the genuses of C and C' respectively, then these numbers are related by the *Riemann–Hurwitz formula*

$$2(g-1) = 2m(g'-1) + \sum_{a \in C} (k_a - 1). \tag{B.1}$$

For any affine curve C the Cartesian projection $\pi : \mathbb{C}^2 \to \mathbb{C}$, $(z, w) \mapsto z$, restricted on C, extends to a holomorphic map between the projective closure of C and the Riemann sphere $\mathbb{P}$. If C is smooth, the number of sheets m of this covering is equal to the degree of C. The genus of the projective line is one. An easy computation of the total ramification index yields the formula $g = \frac{1}{2}(m-1)(m-2)$ for the genus of a smooth algebraic curve of degree m.

B.3. Meromorphic functions on Riemann surfaces. By the maximum modulus principle, there are no holomorphic maps from a compact Riemann surface to $\mathbb{C}$, hence there are no globally defined holomorphic functions. A map $f\colon C \to \mathbb{P}$ is called a *meromorphic function* on C. Locally any meromorphic function can be represented by a ratio of two holomorphic functions. For any point $a \in C$ the *order* $\operatorname{ord}_a f \in \mathbb{Z}$ is an integer number equal to the order of zero or the negative order of pole of f at a. This order is well defined independently of the choice of the local chart on (C, a) used for its computation (we assign the value $\operatorname{ord}_a f = 0$ if a is neither a root nor the pole of f).

Meromorphic functions on the projective line $\mathbb{P}$ are *rational* (i.e., polynomial in z and z^{-1} in the affine chart $\mathbb{C} \subset \mathbb{P}$). Holomorphic functions on an affine algebraic curve that are meromorphic on its closure, are restrictions of polynomials in two variables onto this curve. A meromorphic function on a projective algebraic curve $C \subset \mathbb{P}^2$ is always the restriction of some rational function $P(x, y)/Q(x, y)$ onto this curve.

For any meromorphic function on a compact Riemann surface,

$$\forall f \in \mathfrak{M}(C) \qquad \sum_a \operatorname{ord}_a f = 0. \tag{B.2}$$

B.4. Holomorphic and meromorphic forms on Riemann surfaces. A differential 1-form ω on a Riemann surface C is holomorphic (resp., meromorphic) if in any local chart z it has the form $\omega_z = f(z)\,dz$, where the coefficient f is holomorphic (resp., meromorphic). Poles of the coefficient f are called the poles of the form. The Cauchy-Riemann equation $\frac{\partial \bar z}{\partial =}0$ implies that any holomorphic 1-form on a Riemann surface is closed, $d\omega = 0$. Hence by the Stokes formula the integral of a holomorphic 1-form over a cycle on a Riemann surface depends on the homology class of the cycle only.

In particular, the integral

$$\operatorname{res}_a \omega = \frac{1}{2\pi i} \oint_\gamma \omega \tag{B.3}$$

of a meromorphic 1-form over any small loop around a point $a \in C$, does not depend on the loop and is called *the residue* of the form at a (the residue is zero if ω is holomorphic at a). By the Stokes theorem, for any meromorphic 1-form ω,

$$\sum_{a\in C} \operatorname{res}_a \omega = 0. \tag{B.4}$$

Applied to the logarithmic derivative $\omega = f^{-1}df$ of a meromorphic function $f \in \mathfrak{M}(C)$, this identity implies (B.2).

B.5. Uniformization. There are three examples of simply connected Riemann surfaces that are not pairwise conformally equivalent: an open disc $\mathbb{D} = \{|z| < 1\}$, the complex line $\mathbb{C}$ and the Riemann sphere $\mathbb{P}$. The Poincaré–Kœbe uniformization theorem claims that these are the only possibilities: any simply connected Riemann surface is biholomorphically equivalent either to $\mathbb{D}$, $\mathbb{C}$ or $\mathbb{P}$. This implies, in particular, that any Riemann surface with a cyclic fundamental group is conformally equivalent either to $\mathbb{C}^* = \{0 < |z|\}$, to an annulus $\{\varepsilon < |z| < 1\}$, $\varepsilon > 0$, or to a punctures disc $\{0 < |z| < 1\}$. This trichotomy lies in the background of the study of parabolic germs in Chapter IV.