Problem 19.12. Prove that a linear equation of order n with two regular singularities at $t = 0$ and $t = \infty$ is an Euler equation, i.e., it has the form

$$Lu = 0, \quad L = D^n + a_1 D^{n-1} + \cdots + a_{n-1} D + a_n, \quad a_j \in \mathbb{C},\ D = t\tfrac{\partial}{\partial t}.$$

Find the complete factorization of the Euler equation into composition of first order Fuchsian operators.

Problem 19.13. Prove that for a regular linear equation with m singular points on a compact Riemann surface T, the sum of all characteristic exponents is equal to $(m - \chi)n(n-1)/2$, where $\chi = \deg \mathbf{T}^*T$ is the Euler characteristic (the degree of the cotangent bundle).

Exercise 19.14. Let $D = t\frac{\partial}{\partial t}$ be the Euler operator and u is the "operator of multiplication by t^λ", $\lambda \in \mathbb{C}$. Prove that the conjugated operator $u^{-1}Du$ is again a first order with meromorphic coefficients. Compute it.

Exercise 19.15. Let u be the operator of multiplication by a germ $c(t - t_0)^\lambda h(t)$, $h \in \mathcal{O}(\mathbb{C}, t_0)$, $h(t_0) \neq 0$. Prove that there exists a holomorphic vector field $D \in \mathcal{D}(\mathbb{C}, t_0)$ with a simple (hyperbolic) singular point, such that $u^{-1}Du = D + \lambda$ (cf. with the previous exercise).

Problem 19.16. Show that for an arbitrary Fuchsian operator L of order n with singularities at the points $t_1, \dots, t_m \in \mathbb{P}$ and arbitrary collection of the complex numbers $\lambda_1, \dots, \lambda_m$ such that $\sum \lambda_j = 0$, one can find another Fuchsian operator L' with the same singular points, such that the characteristic exponents $\alpha_{1,j}, \dots, \alpha_{n,j}$ at each singular point t_j are shifted by λ_j: $\alpha'_{i,j} = \alpha_{i,j} + \lambda_j$ for all i, j.

Problem 19.17. Find explicitly the hypergeometric equation (19.30) and the corresponding characteristic exponents for each component of the system (19.34).

20. Irregular singularities and the Stokes phenomenon

Unlike the Fuchsian singularities which can always be reduced to a simple formal normal form by means of a convergent gauge transform, irregular singularities have the formal classification considerably more involved and the normalizing transformations as a rule diverge.

20A. One-dimensional irregular singular points. Irregular singularities of scalar (one-dimensional) linear equations admit complete investigation. Consider the equation

$$t^m \dot{x} = a(t)x, \qquad m \geqslant 2, \quad a(t) = \lambda + a_1 t + a_2 t^2 + \cdots \in \mathcal{O}(\mathbb{C}, 0). \tag{20.1}$$

Its nontrivial solution is given by the explicit formula

$$x(t) = \exp \int \frac{a(t)}{t^m}\, dt = \exp[-t^{1-m}\lambda(1 + o(1))]. \tag{20.2}$$

The origin is an essential singularity of the function $x(t)$ holomorphic in the punctured neighborhood $(\mathbb{C}, 0) \smallsetminus \{0\}$.

Consider $2m-2$ rays from the origin on the complex plane $\mathbb{C}$, described by the condition

$$\operatorname{Re}(\lambda/t^{m-1}) = 0. \tag{20.3}$$

These rays subdivide the neighborhood $(\mathbb{C}, 0)$ into sectors of equal opening $\pi/(m-1)$.

Definition 20.1. An open sector bounded by two rays (20.3) is called the *sector of jump* (resp., *sector of fall*), if the real part of the ratio $\operatorname{Re}(\lambda/t^{m-1})$ is negative (resp., positive) in the interior of this sector.

In each proper subsector of these sectors the solution $x(t)$ of (20.2) grows exponentially fast (in the jump sectors) and is *flat* at $t = 0$ (in the fall sectors). This explains the terminology, as follows from the formula (20.2).

Holomorphic classification of one-dimensional systems is very simple. Clearly, the order m is invariant; the following assertion shows that the $(m-1)$-jet of the coefficient $a(t)$ is a complete invariant of the classification, both formal and holomorphic.

Proposition 20.2. *Two meromorphic one-dimensional "linear systems" (equations) of the form* (20.1) *with the holomorphic coefficients* $a(t), a'(t) \in \mathcal{O}(\mathbb{C}, 0)$, *are holomorphically or formally gauge equivalent if and only the difference* $a(t) - a'(t)$ *is* m*-flat at the origin. In particular, any such equation is equivalent to a unique polynomial equation*

$$t^m \dot{x} = p(t), \qquad p \in \mathbb{C}[t], \quad \deg p \leqslant m-1, \quad p(0) = \lambda. \tag{20.4}$$

Proof. Any conjugacy $x \mapsto h(t)x$ between these equations must satisfy the condition $\dot{h}/h = (a - a')/t^m$ so h is holomorphic and invertible at the origin if and only if the right hand side is holomorphic at the origin. □

20B. Birkhoff standard form. A general (matrix) linear system of any dimension near a non-Fuchsian singular point can be reduced to a polynomial normal form, if the monodromy of the singular point is diagonalizable.

Consider a linear system of the form

$$t^m \dot{X} = A(t)X, \qquad A(t) \in \operatorname{Mat}(n, \mathcal{O}(\mathbb{C}, 0)), \quad A(0) = A_0, \tag{20.5}$$

with the *leading matrix* coefficient $A_0 \in \operatorname{Mat}(n, \mathbb{C})$. Recall that the integer number $m-1$ is the Poincaré rank of the singularity.

Theorem 20.3 (Birkhoff, 1913)**.** *If the monodromy operator* M *of a system* (20.5) *is diagonal(izable), then this system is holomorphically gauge equivalent to a polynomial system*

$$t^m \dot{X} = A_0' + tA_1' + t^2 A_2 + \cdots + t^{m-1} A_{m-1}', \qquad A_i' \in \operatorname{Mat}(n, \mathbb{C}). \tag{20.6}$$

Proof. Let Λ be a diagonal matrix logarithm satisfying the condition $\exp 2\pi i\Lambda = M$. Then any fundamental matrix solution has the form $X(t) = F(t)\, t^{\Lambda}$, where F is a matrix function, single-valued and holomorphically invertible in the punctured neighborhood of the origin but eventually having an essential singularity at $t = 0$.

The function F considered as a Birkhoff–Grothendieck cocycle, is biholomorphically equivalent to a standard cocycle $t^{D'}$ inscribed in a covering

$$U_0 = \{|t| < r_0\},\ U_1 = \{|t| > r_1\}, \qquad U_i \subset \mathbb{P},$$

with sufficiently small values $0 < r_1 < r_0 \ll 1$. In other words, there exist a diagonal integer matrix D' and two holomorphic invertible matrix functions H'_0, H'_1 such that

$$F(t) = H'_0(t)\, t^{D'} H'_1(t), \quad H'_i \in \mathrm{GL}(n, U_i),\ i = 0, 1,\ D' = \mathrm{diag}\{d_1, \dots, d_n\},$$

Using the Permutation Lemma 16.36, we can find a monopole (matrix polynomial with constant nonzero determinant) Π such that $t^{D'} H'_1 = \Pi H_1 t^D$ with $H_1 \in \mathrm{GL}(n, U_1)$ still holomorphic at infinity and D a diagonal matrix obtained by permutation of entries from the diagonal matrix D'. The matrix $H'_0\Pi$ is holomorphic and invertible in U_0. Substituting, we obtain the representation[6]

$$F = H_0 H_1\, t^D, \qquad H_i \in \mathrm{GL}(n, U_i),\ i = 0, 1. \tag{20.7}$$

In fact, the function H_1 and its inverse are holomorphic in $\mathbb{P} \smallsetminus \{0\}$, i.e., both are entire functions of t^{-1}; its extension to the punctured neighborhood of the origin is given by rereading (20.7), $H_1 = H_0^{-1} F\, t^{-D}$.

Since the diagonal matrices Λ and D commute, the solution X of the irregular system can be represented as $X(t) = H_0 \cdot H_1\, t^{\Lambda'}$, $\Lambda' = D + \Lambda$.

After the gauge transform $X \mapsto X' = H_0^{-1} X$ holomorphic at the origin, the logarithmic derivative

$$\Omega' = dX' \cdot (X')^{-1} = dH_1 \cdot H_1^{-1} + t^{-1}\, H_1 \Lambda' H_1^{-1}$$

can be extended on the whole Riemann sphere $\mathbb{P}$. This extension will have a simple pole at infinity and no other singularities except for $t = 0$.

The origin $t = 0$ is a pole of order m for Ω'. Indeed, it was a pole of order m for $\Omega = dX \cdot X^{-1}$; since Ω' and Ω are locally holomorphically conjugate at the origin by construction, this assertion is valid also for Ω'.

Thus the holomorphic gauge transform Ω' of the initial irregular system is a rational matrix 1-form on $\mathbb{P}$ with poles of order m at the origin and 1 at infinity. Thus the matrix coefficient $A'(t)$ of $\Omega' = A'\, dt$ must be a matrix

[6] Sometimes the factorization (20.7) itself is called the Birkhoff factorization of the matrix function F holomorphic in the annulus; see [**FM98**].

polynomial of degree m in t^{-1} without the free term (so that Ω' has at most a simple pole at infinity), exactly as was asserted. □

We wish to stress that Theorem 20.3 is a *global statement*, closely related to Theorem 18.6. If the monodromy is not diagonalizable, then the assertion is in general false [**Gan59**]. However, for *irreducible* irregular singularities the polynomial standard form still exists, as was shown in [**Bol94**]. In fact, this result is closely related to the Bolibruch–Kostov Theorem 18.14.

Recall that a meromorphic connexion (or a linear system) is reducible, if there exists an invariant holomorphic subbundle. Local reducibility means that the invariant subbundles exist locally near a singular point. After rectification of the corresponding subbundles by a suitable holomorphic gauge transform, a locally reducible system can always be brought into block upper-triangular form. A connexion (resp., linear system) is locally irreducible if it admits no nontrivial invariant holomorphic subbundles.

A regular (in particular, Fuchsian) singularity is always locally reducible: the monodromy operator M always has at least one invariant subspace in each dimension, and by Proposition 18.8, each such subspace spans an invariant subbundle. However, for *irregular* singularities Proposition 18.8 in general fails and there exist *locally irreducible singularities* (though this irreducibility is very difficult to check).

Theorem 20.4 (A. Bolibruch, [**Bol94**]). *A locally irreducible irregular singularity is holomorphically equivalent to a polynomial system* (20.6).

The proof of this assertion reproduces the proof of Theorem 18.14 with minimal modifications. The key argument is that a locally irreducible connexion on a holomorphic bundle over $\mathbb{P}$ is always globally irreducible.

Proof. We construct an abstract bundle π_{N} over $\mathbb{P}$ with a meromorphic connexion ∇_{N} on it, which has an irregular singular point at $t = 0$, biholomorphically equivalent to the given singularity $\Omega_0 = t^{-m}(A_0 + A_1 t + \cdots)\,dt$, and a Fuchsian singularity at $t = \infty$ with eigenvalues "well apart". Here $\mathrm{N} = \operatorname{diag}\{\nu_1, \cdots, \nu_n\}$ is a diagonal $n \times n$-matrix with sufficiently fast ascending sequence of integer numbers $\nu_1 \ll \nu_2 \ll \cdots \ll \nu_n$: for our purposes it is sufficient to guarantee that $\nu_{i+1} - \nu_i > (m-1)(n-1)$.

To construct this bundle, we assume that the holonomy operator M is upper-triangular and has an upper-triangular matrix logarithm $A = \frac{1}{2\pi i} \ln M$. Then for any choice of the matrix N the logarithmic derivative $\Omega_\infty = dY \cdot Y^{-1}$, where $Y(t) = t^{\mathrm{N}} t^{A}$, has a Fuchsian singularity at infinity (cf. with (18.8)).

Exactly as in the proof of Theorem 18.14, the two forms Ω_0 on $(\mathbb{C}^1, 0)$ and Ω_∞ on $\mathbb{P} \smallsetminus \{0\}$, considered as connexion forms, define a holomorphic bundle π_{N} and a meromorphic connexion ∇_{N} on it, with only two singularities, one of them Fuchsian. The total order of poles of ∇_{N} is equal to $m + 1$.

If the singularity at the origin is irreducible, then the connexion ∇_{N} is globally irreducible, hence the splitting type $D = \operatorname{diag}\{d_1, \ldots, d_n\}$ of the bundle π_{N} is constrained by the inequality $|d_i - d_j| \leqslant (m-1)(n-1)$ (Problem 18.12, a slightly modified version of Theorem 18.12). Trivializing this bundle and making a suitable monopole transform

Π, we obtain (again exactly as in the proof of Theorem 18.14) a meromorphic connexion on the trivial bundle with an irregular singularity at $t = 0$ and a regular singularity with the fundamental solution $X(t) = G(t)\, t^{D'} t^{\mathrm{N}} t^{A} = G(t)\, t^{D'+\mathrm{N}} t^{A}$. In this expression the matrix function $G \in \mathrm{GL}(n, \mathcal{O}(\mathbb{P}, \infty))$ is holomorphically invertible at infinity, and D' is a diagonal matrix obtained from D by permutation of the diagonal entries. Because of the large gaps between the numbers ν_j, the entries of the diagonal matrix $D' + \mathrm{N}$ are still in the ascending order, hence the logarithmic derivative $dX \cdot X^{-1}$ is Fuchsian. Thus after the trivialization and the monopole gauge transform we obtain a rational matrix 1-form Ω' on $\mathbb{P}$ with a pole of order m at the origin and a simple pole at infinity. This gives the polynomial normal form (20.6). □

Remark 20.5. The "polynomial normal form" (20.6) is in general nonintegrable. Moreover, it is nonlocal: each matrix coefficient A'_k of the normal form depends on the entire series $\sum A_k t^k$ in (20.5). Thus its effectiveness in applications is rather limited.

20C. Resonances and formal diagonalization. The first step in the "genuine" classification of general irregular singularities is the formal classification similar to that described in §16**C** for Fuchsian systems with $m = 1$. Exactly as above, the linear system

$$t^m \dot{x} = A(t)x, \qquad A(t) \in \mathrm{Mat}(n, \mathcal{O}(\mathbb{C}, 0)), \tag{20.8}$$

associated with the matrix equation (20.5), can be reduced to a holomorphic vector field in $(\mathbb{C}^{n+1}, 0)$ corresponding to the "nonlinear" system of differential equations

$$\begin{cases} \dot{x} = A_0 x + tA_1 x + \cdots, & x \in (\mathbb{C}^n, 0), \\ \dot{t} = t^m, & t \in (\mathbb{C}, 0). \end{cases} \tag{20.9}$$

The spectrum of linearization of the system (20.9) at the singular point $(0, 0)$ consists of the zero value $\lambda_0 = 0$ (since $m \geqslant 2$) and the eigenvalues $\lambda_1, \dots, \lambda_n \in \mathbb{C}$ of the leading matrix coefficient $A_0 \in \mathrm{Mat}(n, \mathbb{C})$ (repetitions allowed).

Applying the Poincaré–Dulac technique to the nonlinear system (20.9), we can eliminate from its Taylor expansion all nonresonant terms. Exactly as in the case with Fuchsian systems in §16**C**, only occurrences of *cross-resonances* $\lambda_i = \lambda_j + k\lambda_0$ corresponding to the vector-monomials $t^k x_j \frac{\partial}{\partial x_i}$ will matter. As $\lambda_0 = 0$, this motivates the following definition.

Definition 20.6. The system (20.5) is said to be *nonresonant* at the origin, if all eigenvalues $\lambda_1, \dots, \lambda_n$ of the leading matrix coefficient A_0 are pairwise different.

Theorem 20.7. *A non-Fuchsian system* (20.5) *at a nonresonant singular point* $t = 0$ *is formally gauge equivalent to a diagonal polynomial system of* *degree m,*

$$\begin{aligned} t^m\dot x = \Lambda(t)x, &\qquad \Lambda(t) = \operatorname{diag}\{p_1(t),\dots,p_n(t)\},\\ p_i\in\mathbb{C}[t],\quad \deg p_i = m, &\qquad \Lambda(0) = \operatorname{diag}\{\lambda_1,\dots,\lambda_n\}. \end{aligned} \tag{20.10}$$

Proof. The same (literally) arguments that proved Theorem 16.15 in §16**C**, prove also that only resonant monomials of the form $c_{ijk}t^k x_j \frac{\partial}{\partial x_i}$ should be kept in the expansion (20.9), all others being removable. Elimination of the resonant monomials of degree $k \geqslant m$ can be achieved by Proposition 20.2 and the remark after it. □

As follows from the analysis of the scalar case in §20**A**, a system in the formal normal form (20.10) is integrable: there are diagonal matrix polynomial $B(t^{-1}) = B_0t^{1-m} + B_1t^{2-m} + \cdots + B_{m-2}t^{-1}$ and a constant diagonal matrix C, such that a fundamental matrix solution of (20.5) has the form $X(t) = t^C \exp B(t^{-1})$.

Remark 20.8. Note that the formal series that conjugate irregular singularities may diverge. Indeed, the nonresonant irregular system

$$t^2\frac{d}{dt}\begin{pmatrix} y\\ z\end{pmatrix} = \begin{pmatrix} 1 & -1\\ 0 & t\end{pmatrix}\begin{pmatrix} y\\ z\end{pmatrix}, \tag{20.11}$$

with a separating second variable can be reduced to the Euler equation (7.11) (Example 7.10). The Euler equation has a formal Taylor solution which diverges. Clearly, this would be impossible were the normalizing series convergent.

20D. Formal simplification in the resonant case. The direct proof of the formal diagonalization Theorem 20.7 looks as follows. The formal gauge transformation $X \mapsto X' = HX$ defined by a formal matrix series

$$H = E + \sum_{k>0} t^k H_k \in \mathrm{GL}(n, \mathbb{C}[[t]])$$

conjugates two systems (formal or convergent)

$$\begin{aligned} t^m\dot X &= A(t)X, & & & t^m\dot X' &= A'(t)X',\\ A(t) &= A_0 + \sum_{k>0} t^k A_k, &\quad &\text{and}\quad & A'(t) &= A_0 + \sum_{k>0} t^k A'_k, \end{aligned}$$

with the same principal part $A(0) = A'(0) = A_0$, if and only if H is a formal solution to the following matrix differential equation,

$$t^m\dot H = A'(t)H - HA(t). \tag{20.12}$$

Termwise, this equation is equivalent to the sequence of matrix equations involving the coefficients A_k, A'_k of the expansions for $A(t)$ and $A'(t)$ respectively,

$$\begin{aligned} 0 &= (A'_0H_k - H_kA_0) + (A'_k - A_k)\\ &+ \sum_{i,j>0,\ i+j<k} (A'_iH_j - H_iA_j) - \begin{cases} kH_{k+1-m}, & k \geqslant m-1,\\ 0, & k < m-1. \end{cases} \end{aligned} \tag{20.13}$$

These equations can be rewritten in the form

$$[A_0, H_k] + A'_k = \text{matrix polynomial in } \{A'_j, H_j,\ 0 < j < k\}.$$

By Lemma 4.11, the image of the operator $\operatorname{ad}_{A_0}\colon B \mapsto [A_0, B]$ is a linear subspace in $\operatorname{Mat}(n, \mathbb{C})$ orthogonal (in the sense of some Hermitian structure) to the subspace of all matrices commuting with the conjugate matrix A_0^*. Thus the equations (20.13) are always solvable for suitable matrices H_k and A'_k such that $[A_0^*, A_k] = 0$.

If A_0 is nonresonant, it can be diagonalized, $A_0 = \Lambda = \operatorname{diag}\{\lambda_1, \dots, \lambda_n\}$, so that $\operatorname{Ker}\operatorname{ad}_{\Lambda^*}$ consists of diagonal matrices only. Thus a nonresonant irregular singularity is formally diagonalizable. Slightly more generally, if A_0 is block diagonal with each block having only one eigenvalue different for different blocks, then the complementary subspace can be chosen as matrices having the same block diagonal structure. This proves the following generalization of Theorem 20.7.

Theorem 20.9. *By a formal gauge transformation one can reduce an irregular system to the block-diagonal form with each block having the leading matrix with a single eigenvalue.*

Example 20.10. Assume that the leading matrix A_0 is a single Jordan block of size n with the eigenvalue λ_0, $A_0 = \lambda_0 E + J$. For an arbitrary matrix B commutation with J^* means that shifts of the columns of B to the left and shift of its rows downward produce the same result (in both cases the null column or row is added). Thus for any element B_{ij} the elements next to the right and one row above it coincide, the elements of the first row and the last column being all zeros. Thus $[B, J^*] = 0$, if and only if all elements on each secondary diagonal (parallel to the principal diagonal) are equal among themselves and equal to zero in the upper-right half (so that B is lower triangular).

Thus an irregular singularity with the leading matrix coefficient $A_0 = \lambda_0 + J$ can be brought to the form (20.8) in which

$$A(t) = \begin{pmatrix} \lambda_0 & 1 & & & \\ b_1(t) & \lambda_0 & 1 & & \\ \dots & \dots & \dots & \dots & \dots \\ b_{n-2}(t) & \cdots & b_1(t) & \lambda_0 & 1 \\ b_{n-1}(t) & b_{n-2}(t) & \cdots & b_1(t) & \lambda_0 \end{pmatrix}.$$

In fact, one can further simplify the obtained normal form and get rid of all entries except those in the last row; see [**Arn83**, §30]. As a result, by a formal gauge transformation the system is reduced to the companion form modulo a scalar matrix,

$$A(t) = \lambda_0 E + \begin{pmatrix} 0 & 1 & & & \\ & 0 & 1 & & \\ \dots & \dots & \dots & \dots & \dots \\ & & & 0 & 1 \\ a_n(t) & a_{n-1}(t) & \cdots & a_2(t) & a_1(t) \end{pmatrix} \tag{20.14}$$

with formal series $a_i \in \mathbb{C}[[t]]$. The eigenvalues of the matrix $A(t)$ are of the form $\lambda_0 + \lambda_i(t)$, where $\lambda_1(t), \dots, \lambda_n(t)$ are the roots of the characteristic equation

$$\lambda^n = a_1(t)\lambda^{n-1} + \cdots + a_{n-1}(t)\lambda + a_n(t).$$

Since $\lambda_1(0) = \cdots = \lambda_n(0) = 0$ by assumption, we see that the formal series $a_i \in \mathbb{C}[[t]]$ are all without the free terms.

Remark 20.11. If $f(t) = \exp(m\lambda_0/t^{m-1})$ is a solution of the equation $\dot f = -\lambda_0 t^{-m} f$, then the gauge transformation $X \mapsto f(t)X$ brings the system (20.14) to the true companion form (without the diagonal term $\lambda_0 E$). Being scalar, this transformation commutes with any other gauge equivalence, formal or convergent.

20E. Shearing transformations and ramified formal normal form. Further simplification of the system is possible only if we extend the class of formal gauge transformations, allowing for *ramified formal transformations* which are formal series in fractional powers of t. It was E. Fabry who realized (1885) the necessity of passing to fractional powers.

Example 20.12 (continuation of Example 20.10)**.** Consider again the case of a system whose leading matrix is a maximal size Jordan block. By Remark 20.11, without loss of generality we may assume that $\lambda_0 = 0$. Assume that $r \in \mathbb{Q}$ is a *positive* rational number, and consider the gauge transformation

$$H(t) = \operatorname{diag}\left\{1, t^{-r}, t^{-2r}, \dots, t^{(1-n)r}\right\}. \qquad (20.15)$$

This transformation takes the system (20.5) with the matrix $A(t)$ as in (20.14), into that with the matrix

$$\begin{pmatrix} 0 & t^r & & & \\ & 0 & t^r & & \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ & & & 0 & t^r \\ t^{(1-n)r}a_n & t^{(2-n)r}a_{n-1} & \cdots & t^{-r}a_2 & a_1 \end{pmatrix} - t^{m-1}R,$$

where $R = \operatorname{diag}\{0, r, 2r, \dots, (n-1)r\}$ is the diagonal matrix. The orders of zeros $\nu_k \in \mathbb{N}$ of the formal series $a_k(t)$ were all positive, since $a_k(0) = 0$. Choose r so that the orders of all terms $a'_k(t) = t^{-kr}a_k(t)$ are still nonnegative but the smallest of them is zero, $r = \min_k \nu_k/k$. The denominator of r is no greater than n.

After the conjugacy by H the matrix of the system will take the form

$$\dot{X} = [t^{-m+r}A'(t) + t^{-1}R]X, \qquad r > 0, \qquad (20.16)$$

where $A'(t)$ is a companion matrix similar to (20.14) but with the entries $a'_k(t) \in \mathbb{C}[[t^{1/q}]]$, $k = 1, \dots, n$, now being formal series in *fractional* powers of t (and without the diagonal term λ_0). The leading (matrix) coefficient $A'(0)$ of $A'(t)$ is the companion matrix containing the complex numbers $a'_n(0), \dots, a'_1(0)$ as the last row. By the choice of r, *not all of them are simultaneously zero*, yet their sum is zero, since $\operatorname{tr} A'(0) = a'_1(0) = a_1(0) = 0$. Therefore if after the shearing transformation the system remains non-Fuchsian (i.e., if $r < m - 1$), at least some of the leading eigenvalues must be nonzero.

Somewhat more elaborate computations allow us to prove similar statement also in the case where the leading matrix coefficient A_0 has several Jordan blocks with the common eigenvalue.

Notice now that the construction described in §20**D**, applies without any changes to the *ramified* formal series in fractional powers of t (i.e., when the indices i, j, k range over an arithmetic progression with rational noninteger difference). Applying Theorem 20.7 in these extended settings, we see that the system (20.16) can now be formally split into two subsystems.

By iteration of these two steps (splitting the system and subsequent shearing transformation) sufficiently many times, one can prove the following result.

Theorem 20.13 (Hukuhara (1942), Turritin (1955), Levelt (1975))**.** *By a suitable formal ramified gauge transformation an irregular singularity can be reduced to the diagonal form*

$$A(t) = t^{-r_1}P_1 + t^{-r_2}P_2 + \cdots + t^{-r_k}P_k + t^{-1}C,$$

where $r_1 > r_2 > \cdots > r_k > 1$ are rational numbers with the denominators not exceeding $n!$ and $P_1, \dots, P_k \in \operatorname{Mat}(n, \mathbb{C})$ are diagonal matrices commuting with C.

We will not give the proof in full details; see [**Var96**] and the references therein. Instead, we focus on the more transparent nonresonant case and study the problems of *holomorphic* rather than formal classification.

20F. Holomorphic sectorial normalization. Even in the nonresonant case there is a wide gap between formal and analytic classification. In this section we explain the geometric obstructions for convergence of formal normalizing transformations.

Definition 20.14. A *separation ray*[7] corresponding to a fixed value of m and a pair of complex numbers $\lambda \neq \lambda' \in \mathbb{C}$ is any of the $2(m-1)$ rays defined by the condition

$$\operatorname{Re}[(\lambda - \lambda')/t^{m-1}] = 0. \tag{20.17}$$

The following property is characteristic for separation rays, being an immediate consequence of the explicit formula (20.2). Consider two solutions $x(t), x'(t)$ of two scalar systems (20.1) with the same order m and the holomorphic coefficients $a(t), a'(t)$. Denote $\lambda = a(0)$, $\lambda' = a'(0)$. Recall that a function defined and holomorphic in a sector with the vertex at the origin is said to be *flat*, if it decreases faster than any power of the distance to this point, and the same is true for all its derivatives. A reciprocal $1/f$ of a flat nonvanishing function is called *vertical*.

Proposition 20.15. *If $R = \rho \cdot \mathbb{R}_+$, $|\rho| = 1$, is* not *a separation ray for the pair λ, λ', then out of the two reciprocal ratios $x(t)/x'(t)$ and $x'(t)/x(t)$ one after restriction on R is flat and the other is vertical, depending on whether $(\lambda - \lambda')/\rho^{m-1}$ is respectively negative or positive.* □

Everywhere here and below we always assume that any sector is bounded by two straight rays coming from the vertex (usually the origin); the angle between these rays is the *opening* of the sector. If $\widehat{H} \in \mathrm{GL}(n, \mathbb{C}[[t]])$ is a formal power series, we say that a holomorphic matrix function $H \in \mathrm{GL}(n, \mathcal{O}(S))$ *extends* this series, if $\widehat{H}$ is the *asymptotic series* for H in S, that is, the difference between $H(t)$ and any truncation $\widehat{H}_N(t) \in \mathrm{Mat}(n, \mathbb{C}[t])$ of $\widehat{H}$, the matrix polynomial of degree N, decreases faster than t^N,

$$\|H(t) - \widehat{H}_N(t)\| = o(|t|^N) \quad \text{as} \quad t \to 0,\ t \in S, \qquad \forall N \in \mathbb{N}.$$

Theorem 20.16 (sectorial normalization theorem, Y. Sibuya [**Sib62**])**.** *Assume that the leading matrix A_0 of the linear system* (20.5) *is nonresonant* (*i.e., has pairwise different eigenvalues $\lambda_1, \dots, \lambda_n$*)*.*

If $S \subset (\mathbb{C}, 0)$ is an arbitrary sector not containing two separation rays for any pair of the eigenvalues λ_i, λ_j, then any formal gauge transformation $\widehat{H}(t) \in \mathrm{GL}(n, \mathbb{C}[[t]])$ conjugating (20.5) *with its polynomial diagonal normal form* (20.10)*, can be extended to a holomorphic conjugacy $H_S(t) \in \mathrm{GL}(n, \mathcal{O}(S))$ between these systems in S.*

[7]The union of two separating rays in opposite directions is called a *Stokes line* in some sources.

The proof of this theorem is moved to the appendix; see §20**J** below. It differs both from the author's proof in [**Sib90**] and from that in [**Was87**].

20G. Sectorial automorphisms and Stokes matrices. If the sector is sufficiently wide, then the normalizing transform is necessarily unique. This can be seen by studying *automorphisms* of the system in the diagonal normal form. We will show that such systems admit no nontrivial automorphisms over such sectors.

More specifically, assume that $H'(t), H''(t)$ are two sectorial automorphisms conjugating an irregular singularity (20.5) with its diagonal formal normal form (20.10) in some sector $S \subset (\mathbb{C}, 0)$. Then the "superpositional ratio", the sectorial gauge transform with the matrix function $H(t) = H''(t) \cdot {H'}^{-1}(t)$, is an automorphism of the diagonal system (20.10).

Such automorphisms are most easily described by their action on a suitably chosen fundamental solution. In our case the diagonal system (20.10) admits a distinguished set of solutions which are themselves diagonal.

We fix a diagonal fundamental solution $W(t) = \operatorname{diag}\{w_1(t), \dots, w_n(t)\}$ for (20.10). Then any holomorphic sectorial automorphism $H(t)$ of the diagonal normal form, $H \in \mathrm{GL}(n, \mathcal{O}(S, 0))$, is uniquely determined by a constant matrix $C \in \mathrm{GL}(n, \mathbb{C})$ such that

$$H(t)W(t) = W(t)C. \tag{20.18}$$

This matrix will be referred to as the *Stokes matrix* of the sectorial automorphism $H(\cdot)$. This matrix depends on the choice of the diagonal fundamental solution W, yet because of the special growth pattern of solutions it can be rather accurately described.

Lemma 20.17. *Suppose that none of the two rays bounding a sector S is a separation ray for the system* (20.10) *in the formal normal form, and the eigenvalues of the leading matrix Λ_0 are ordered so that* $\operatorname{Re}\lambda_1 < \cdots < \operatorname{Re}\lambda_n$.

Then the Stokes matrix $C \in \mathrm{GL}(n, \mathbb{C})$ of any sectorial automorphism $H \in \mathrm{GL}(n, \mathcal{O}(S, 0))$ which is 0*-tangent to the identity, $H(t) = E + o(1)$, possesses the following properties*:

(1) *For any pair $i \neq j$ of indices, one of the matrix elements c_{ij}, c_{ji} must be zero, in particular,*
(2) *if $S \supset \mathbb{R}_+$, then $C - E$ is a nilpotent upper-triangular matrix.*
(3) *If S contains a separation ray for the pair $\lambda_i \neq \lambda_j$ then both $c_{ij} = c_{ji} = 0$, in particular,*
(4) *if S contains one separation ray for each pair of eigenvalues, then necessarily $C = E$.*

Proof. All assertions immediately follow by inspection of the asymptotic behavior of the sectorial automorphism written in terms of the Stokes matrix,

$$H(t) = W(t)CW^{-1}(t) = \|h_{ij}(t)\|, \qquad h_{ij}(t) = c_{ij}\, w_i(t)/w_j(t),$$

and the observation in Proposition 20.15.

Indeed, if the ratio $w_i(t)/w_j(t)$ along some ray in S is vertical, the corresponding coefficient c_{ij} must necessarily be zero. This proves the first two assertions.

To prove the remaining assertions, note that the two reciprocal ratios w_i/w_j and w_j/w_i have reciprocal asymptotical behavior along any two rays sufficiently close but separated by the separation ray for the eigenvalues λ_i and λ_j. By the preceding arguments, in this case both c_{ij} and c_{ji} must be absent. □

Proposition 20.18 (rigidity). *If a sector S has an opening bigger than $\pi/(m-1)$, then the sectorial normalization H_S described in Theorem 20.16, is unique.*

Proof. If there were two sectorial normalizations H', H'' with the same asymptotic series $\widehat{H}$, then their matrix ratio $H = H''H'^{-1}$ must be a sectorial automorphism of the formal normal form (20.10), tangent to the identity (i.e., of the form id +flat function). Since all separation rays for each pair of eigenvalues are separated by the angle $\pi/(m-1)$, the sector S of opening bigger than $\pi/(m-1)$ must contain at least one such ray for each pair. By the last assertion of Lemma 20.17, the corresponding Stokes matrix must be identity, which means that the ratio itself is identity. □

20H. Stokes phenomenon. Holomorphic classification of irregular singularities. Consider a linear system (20.5) of Poincaré rank $m-1$ at the nonresonant non-Fuchsian singular point $t = 0$, and let (20.10) be its formal normal form.

As before, we can assume without loss of generality that the leading matrix has eigenvalues ordered so that

$$\operatorname{Re}\lambda_1 < \cdots < \operatorname{Re}\lambda_n, \tag{20.19}$$

which means that neither the positive semiaxis $\mathbb{R}_+$ nor its rotated copies $\rho^k\,\mathbb{R}_+$, $k = 1, \dots, 2(m-1)$, where $\rho = \exp\frac{\pi i}{m-1}$, are separation rays for any two eigenvalues $\lambda_i \neq \lambda_j$.

The open sector S^* bounded by the rays $\mathbb{R}_+$ and $\rho\,\mathbb{R}_+$ of opening $\pi/(m-1)$ contains exactly one separation ray for each pair, none of them on the boundary. Thus one can enlarge slightly the opening of this sector to become $2\delta + \pi/(m-1)$ so that it still contains exactly one separation ray for each

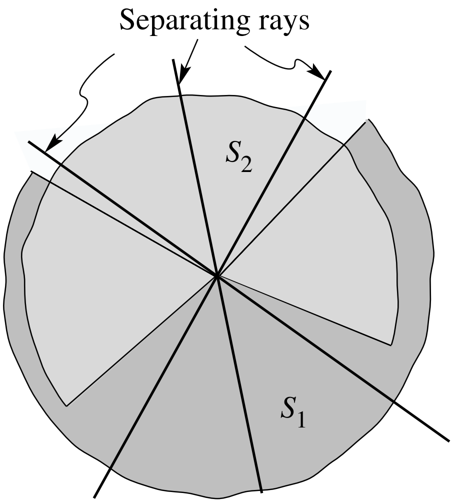

Figure III.3. Standard covering and separation rays in the simplest case $m = 2$

pair. Denote this enlarged sector by $S_1 = \{-\delta < \operatorname{Arg} t < \pi/(m-1)+\delta\}$, and let $S_2, \dots, S_{2(m-1)}$ be its rotated copies, $S_k = \rho^{k-1} S_1$. These sectors form a covering of the punctured neighborhood of the origin; the intersections are narrow flaps $S_{j,j+1} = \{|\operatorname{Arg} t - j\pi/(m-1)| < \delta\}$ of opening $2\delta > 0$ each. This collection of sectors will be referred to as the *standard covering* of the punctured neighborhood of the origin.

By Theorem 20.16, over each sector S_k there exists a holomorphic gauge conjugacy $H_k(t) \in \mathrm{GL}(n, \mathcal{O}(S_k))$ between the initial system (20.5) and its formal normal form (20.10). This conjugacy is unique by Proposition 20.18. The collection $\{H_k\}$ of these sectorial normalizing maps is called the *normalizing cochain* inscribed in the standard covering $\{S_k\}$.

Since all maps forming the normalizing cochain have the same common asymptotic series, the matrix ratios $F_{ij} = H_i H_j^{-1} = F_{ji}^{-1}$ defined on the nonempty intersections $S_i \cap S_j$, are sectorial automorphisms of the formal normal form (20.10). Clearly, the intersections $S_i \cap S_j$ are nonvoid if and only if $j = i + 1$ cyclically modulo $2(m-1)$; they are thin sectors around the rotated copies $\rho^j \,\mathbb{R}_+$ of the real axis.

Let $\{H_i\}$ be the (uniquely defined) normalizing cochain inscribed in the standard covering. Choose a diagonal fundamental matrix solution $W(t)$; since in general the normal form has a nontrivial monodromy, the solution $W(t)$ is multivalued. To avoid this, we slit the neighborhood along the ray $\{\operatorname{Arg} t = \pi/2(m-1)\}$ entirely belonging to the sector S_1 and disjoint with all overlapping sectors $S_{ij} = S_i \cap S_j$, $|i-j| = 1$, and consider a fundamental solution in the slit domain. Such a solution is defined uniquely modulo a diagonal transform

$$W(t) \mapsto DW(t) = W(t)D, \qquad D = \operatorname{diag}\{\alpha_1, \dots, \alpha_n\}, \tag{20.20}$$

and by construction it is holomorphic in all flaps S_{ij}.

Definition 20.19. The *Stokes collection* of a linear system at a nonresonant irregular singular point is the collection of Stokes matrices $\{C_j\}$, $j = 1, \dots, 2(m-1)$ of the sectorial automorphisms $F_{ij} = H_i H_j^{-1}$, $i+1 = j$, corresponding to a diagonal solution $W(t)$ of the formal normal form.

Proposition 20.20. *The matrices C_j from the Stokes collection are unipotent.*

Proof. If S is a sector containing the positive semiaxis and the eigenvalues of Λ_0 are ordered as in (20.19), the assertion follows from the second assertion of Lemma 20.17. The general case can be brought to the former specific case by suitable rotation of the t-plane and re-enumeration of the eigenvalues. ☐

By Proposition 20.18, the Stokes collection is uniquely defined, as soon as the diagonal fundamental solution $W(t)$ is fixed. Replacing the diagonal solution $W(t)$ by another solution $DW(t) = W(t)D$ transforms the Stokes matrices by the simultaneous diagonal conjugacy

$$\begin{aligned} C_j \mapsto C_j' &= DC_jD^{-1}, \\ D &= \operatorname{diag}\{\alpha_1, \dots, \alpha_n\}, \end{aligned} \qquad \forall j = 1, \dots, 2(m-1). \tag{20.21}$$

The Stokes collections $\{C_1, \dots, C_{2m-2}\}$ and $\{C_1, \dots, C_{2m-2}\}$ related by the transformation (20.21), are called *equivalent* Stokes collections. Note that the *trivial* collection $C_1 = \cdots = C_{2m-2} = E$ is equivalent only to itself.

Theorem 20.21 (classification theorem for nonresonant irregular singularities)**.** *Any two nonresonant irregular linear systems with a common formal normal form are locally holomorphically gauge equivalent if and only if their Stokes collections are equivalent in the sense* (20.21).

In particular, a linear system is holomorphically equivalent to its formal normal form, if and only if the Stokes collection is trivial.

Proof. Consider two systems with the same formal normal form. Without loss of generality we may assume that a common standard covering is chosen, and the uniquely defined normalizing cochains are denoted by $\{H_j\}$ and $\{H_j'\}$ respectively.

Let G be a holomorphic conjugacy between these systems. Together with the cochain $\{H_j'\}$, the cochain $\{H_jG\}$ clearly is also a normalizing cochain for the second system. By the uniqueness (Proposition 20.18), $H_j' = H_jG$ and hence $H_i'(H_j')^{-1} = DH_iH_j^{-1}D^{-1}$ for all $|i-j| = 1$. Coincidence of the transition cocycles means that the corresponding Stokes collections (apriori defined with respect to two different fundamental solutions W and $W' = DW$) are equivalent.

In the inverse direction this argument also works. If two Stokes collections are equivalent, then by choosing another diagonal fundamental solution we can guarantee that the corresponding Stokes operators simply coincide. Then the matrix quotients $G_j = H'_j H_j^{-1}$ and $G_i = H'_i H_i^{-1}$ coincide on the nonvoid intersections (when $|i-j|=1$) and hence together define a matrix function G holomorphically invertible outside the origin. This function has an asymptotic series equal to the matrix ratio of two formal normalizing gauge transforms $\widehat{H}'\widehat{H}^{-1}$ for the two systems, hence extends at the origin. □

20I. Realization theorem. Proposition 20.20 describes the necessary property of Stokes operators associated with the given order m and a collection of eigenvalues $\lambda_1, \dots, \lambda_n$. It turns out that this is a unique requirement.

Theorem 20.22 (Birkhoff, 1909)**.** *Any collection of unipotent upper-triangular matrices $\{C_i\}$ meeting the restrictions from Proposition 20.20, can be realized as the Stokes collection of a nonresonant irregular singularity with a preassigned formal normal form* (20.10)*.*

Sketch of the proof. Consider the diagonal formal normal form (20.10), the standard covering S_j and the collection of holomorphic invertible matrix functions

$$F_{j,j+1}(t) = W(t)C_jW^{-1}(t) = F_{j+1,j}^{-1}(t), \qquad j = 1, \dots, 2(m-1),$$

defined in the corresponding nonempty intersections $S_{ij} = S_i \cap S_j$, $|i-j| = 1$. Here $W(t)$ is a diagonal fundamental solution of the formal normal form, holomorphic in the small neighborhood of the origin $(\mathbb{C}, 0)$ slit along the ray $\{\operatorname{Arg} t = \pi/2(m-1)\} \subset S_1$ as before. By our assumptions, the constant matrices C_j are related to the eigenvalues λ_j in such a way that the differences $F_{ij}(t) - E$ are flat in the thin sectors S_{ij}.

It can be shown that the cocycle $\mathcal{F} = \{F_{ij}\}$ is solvable by a holomorphic cochain $\mathcal{H} = \{H_j\}$ of holomorphic invertible matrix functions so that $F_{ij}H_j = H_i$ for $|i-j| = 1$. This means that the sectorial solutions $X_j(t) = H_j^{-1}(t)W(t) = X_i(t)C_j$ satisfy linear systems with the coefficient matrices

$$A_j(t) = t^m \tfrac{d}{dt}(H_j^{-1})H_j + H_j^{-1}(t)\varLambda(t)H_j(t)$$

coinciding on the intersections, $A_i(t) = A_j(t)$ for $t \in S_i \cap S_j$. The resulting matrix function $A(t)$, defined in the punctured neighborhood of the origin, is bounded hence holomorphic and by construction the system $t^m\dot{X} = A(t)X$ is holomorphically equivalent to the formal normal form $t^m\dot{W} = \varLambda(t)W$.

Clearly, the Stokes collection of the constructed system coincides with the prescribed data $\{C_j\}$.

Geometrically this construction consists of patching together linear systems defined over different sectors S_j, using the gauge maps F_{ij}, $|i-j| = 1$, for identification. The result will be a linear system defined on a holomorphic vector bundle over the punctured neighborhood $(\mathbb{C}, 0) \setminus \{0\}$. Such a bundle is always holomorphically trivial, as any bundle over a noncompact Riemann surface [**For91**, §30]. The delicate circumstance is to verify that the linear system which appears after trivialization of this bundle, will have an irregular singularity of the prescribed formal type. The solvability of the "asymptotically trivial" cocycle $\{F_{ij}\}$ by a *holomorphic* cochain $\{H_i\}$ guarantees this automatically. Details can be found in [**BV89**]. □

As a corollary we conclude that there exist non-Fuchsian systems for which the formal diagonalizing series diverge. Moreover, in some sense this divergence is characteristic for the *majority* of non-Fuchsian singularities: Theorems 20.21 and 20.22 imply that classes of holomorphic gauge equivalence are parameterized by $(m-1)n(n-1)$ complex parameters (entries of the Stokes collections).

Appendix: Demonstration of Sibuya theorem

In this section we prove the Sectorial Normalization Theorem 20.16. This theorem can be reduced to an analytic claim asserting existence of flat solutions for a nonhomogeneous system of linear equations in a sector.

Throughout this appendix we fix a nonresonant linear system (20.5), its diagonal formal normal form (20.10) with $\Lambda(0) = \operatorname{diag}\{\lambda_1, \dots, \lambda_n\}$, $\lambda_i \neq \lambda_j$, and a formal transformation $\widehat{H} \in \mathrm{GL}(n, \mathbb{C}[[t]])$ conjugating the two. Given a sector S, we can then speak about sectorial conjugacy (or conjugacies) extending $\widehat{H}$ in this sector.

20J. Normalization in "acute" sectors. First we show that the problem of constructing holomorphic sectorial normalization conjugating an irregular singularity with its diagonal formal normal form, can be solved in any sufficiently "acute" sector, namely, if the opening of this sector is less than $\pi/(m-1)$. Enlarging this sector to wider sectors S_j of opening $\pi/(m-1)+2\delta$ forming the standard covering, is achieved relatively simply in §20**M**.

By the Borel–Ritt theorem [**Was87**, §9.2] (see also Problem 20.2), in any sector S there exists an analytic matrix function $F(t)$ whose asymptotic series in S is the prescribed normalizing series $\widehat{H}$. Conjugating the system (20.5) by F, we obtain a new system of the form $t^m \dot{X} = A'(t)X$ with the matrix $A'(t)$ holomorphic in S and having the same asymptotic series at the origin as the Taylor series $\Lambda(t)$ of the formal normal form $t^m\dot{X} = \Lambda(t)X$. Thus to construct the sectorial conjugacy between the system and its initial normal form, it is sufficient to remove by a suitable sectorial gauge transformation the *flat* nondiagonal part $B(t)$ from the system

$$\begin{gathered} t^m\dot{X} = (\Lambda(t) + B(t))X, \qquad B(t) = \|b_{ij}(t)\|, \\ b_{ij} \in \mathcal{O}(S), \quad b_{ii} \equiv 0, \quad b_{ij} \text{ flat in } S, \\ S = \{\alpha < \operatorname{Arg} t < \beta,\ |t| < r\}, \qquad |\beta - \alpha| = \pi/(m-1) - 2\delta. \end{gathered} \tag{20.22}$$

The diagonal entries of B can be assumed absent by Proposition 20.2. The positive parameters $1 \gg \delta > 0$ and $0 < r \ll 1$ characterizing the sector S, can be assumed as small as necessary.

A conjugacy $H(t)$ between (20.22) and (20.10), holomorphic in the sector S with the identical asymptotic series, satisfies the differential equation

$$t^m \dot{H} = \Lambda H - H(\Lambda + B) = [\Lambda, H] - HB. \tag{20.23}$$

The flat difference $Y(t) = H(t) - E$ satisfies the equation

$$\dot{Y} = [\Lambda, Y] - (E + Y)B, \qquad t \in S, \ B(\cdot) \text{ flat in } S. \tag{20.24}$$

Denote by $y = (y_1, \dots, y_k) \in \mathbb{C}^k$, $k = n(n-1)/2$, the collection of all off-diagonal entries of the matrix Y. The system (20.24) then takes the form

$$t^m \dot{y}(t) = [D + G(t)]y(t) + g(t), \qquad t \in S, \tag{20.25}$$

where D is a diagonal matrix corresponding to the commutator with the leading term $\Lambda_0 = \operatorname{diag}\{\lambda_1, \dots, \lambda_n\}$ of the formal normal form $\Lambda(t)$,

$$D \colon Y \mapsto DY = [\Lambda_0, Y].$$

Since the system was assumed nonresonant, all eigenvalues of D are nonzero,

$$D = \operatorname{diag}\{\mu_1, \dots, \mu_k\}, \qquad \mu_i \neq 0, \ i = 1, \dots, k, \ k = n(n-1)/2. \tag{20.26}$$

The term $G(t)$ corresponds to the commutator with the nonleading terms and the multiplication by the flat off-diagonal terms from the matrix B,

$$Y \mapsto GY = [\Lambda(t) - \Lambda_0, Y] + YB(t).$$

In our assumptions $G(t)$ tends to zero as $t \to 0$. The nonhomogeneity $g(t)$ consists of the off-diagonal terms of the matrix $B(t)$ and is also flat at the origin.

It is convenient to simplify the system further to reduce the Poincaré rank to the minimum and place the singular point at infinity so that the leading part would be a system with constant coefficients easy for explicit integration.

Changing the independent variable from $t \in S \subset (\mathbb{C}, 0)$ to $z = 1/t^{m-1} \in (\mathbb{P}, \infty)$ transforms the 1-form $t^{-m}\,dt$ to $(1-m)\,dz$. This transformation brings the system (20.25) to the form $dy/dz = (1-m)[D + G(z^{1/(1-m)})] + (1-m)g(z^{1/(1-m)})$ defined in a sector S' with the vertex at infinity and the opening strictly less than π. Rotating the z-plane if necessary, we can always assume that $S' = \{|z| > r, \ |\operatorname{Arg} z| < \frac{\pi}{2} - \delta\}$, where $\delta > 0$ is a small positive parameter.

Returning to the previous notations, we can rewrite the system (20.25) with respect to the new variable z as follows:

$$\begin{gathered}\tfrac{d}{dz}y = [D + G(z)]y + g(z), \qquad y \in \mathbb{C}^k,\\ z \in S' = \{|z| > r,\ |\operatorname{Arg} z| < \tfrac{\pi}{2} - \delta\},\\ G(z) = o(1),\ g(z) = o(z^{-N}) \quad \text{for any } N \in \mathbb{N}, \quad \text{as } z \xrightarrow[S']{} \infty,\\ D = \operatorname{diag}\{\mu_1, \dots, \mu_k\}, \quad \mu_i \neq 0.\end{gathered} \tag{20.27}$$

Theorem 20.23. *The system* (20.27) *admits a flat solution holomorphic in the sector* S'.

Corollary 20.24. *The system* (20.24) *admits holomorphic flat solution* $Y \in \mathcal{O}(S)$ *in any "acute" sector* S *of opening less than* $\pi/(m-1)$.

The key idea of the proof of this theorem is to treat the system (20.27) as a perturbation of the linear diagonal equation

$$\frac{dy}{dz} = Dy, \qquad z \in S',\ D = \operatorname{diag}\{\mu_1, \dots, \mu_k\}.$$

Since the latter system is immediately integrable, we can explicitly describe the resolvent operator $\mathbf{S}$ for the corresponding nonhomogeneous equation,

$$\frac{dy}{dz} = Dy + h \iff y = \mathbf{S}h,$$

by the method of variation of constants. The resolvent $\mathbf{S}$ turns out to be a bounded linear integral operator for a suitable choice of the paths of integration, as explained in §20**K**. Using the resolvent $\mathbf{S}$, the initial equation (20.27) can be rewritten as a fixed point equation,

$$y = \mathbf{S}[Gy + g],$$

with the operator $y \mapsto \mathbf{G}y = Gy + g$ so strongly contracting that the composition $\mathbf{SG}$ is a contracting operator on a suitable Banach space.

Now we proceed with a detailed exposition.

20K. Core example. Consider first the particular one-dimensional case of the system (20.27),

$$\tfrac{d}{dz}y = \mu y + g(z), \qquad 0 \neq \mu \in \mathbb{C}, \quad y \in \mathbb{C}^1, \quad z \in S'. \tag{20.28}$$

with a flat nonhomogeneity $g(z) \in \mathcal{O}(S')$ and absent linear nonautonomous term, i.e., $G \equiv 0$. We are looking for a solution flat in the sector S'.

The solution of this system is given by the explicit formula obtained by variation of constants method (see Remark 15.6): for an arbitrary choice of the base point $b \in S'$,

$$y(z) = e^{\mu z}\left(y(b) + \int_b^z e^{-\mu\zeta} g(\zeta)\,d\zeta\right) = e^{\mu z}y(b) + \int_b^z e^{\mu(z-\zeta)} g(\zeta)\,d\zeta. \tag{20.29}$$

The upper limit of integration is the variable point z. The lower limit $b \in S'$ and the respective boundary condition $y(b)$ have to be chosen so that the solution (20.29) would be flat in S'.

Two cases have to be treated separately, depending on the relative position of $0 \neq \mu \in \mathbb{C}$ and S', namely,

(1) $\operatorname{Re} \mu a > 0$ for some $a \in S'$, that is, the solution of the homogeneous equation is unbounded in S'; this happens when S' overlaps with some sector of jump (in the sense of §20**A**), and

(2) $\operatorname{Re} \mu z < 0$ for all $z \in S'$, that is, the solution of the homogeneous equation decays exponentially fast in S' (i.e., when S' belongs to a fall sector).

The intermediate case where $\operatorname{Re} \mu z = 0$ along one of the boundary rays of S', will not be discussed, as we will not need it. We will refer to the sector of the first type as a *mixed sector with the growth direction* $a \in \mathbb{C}$, while calling the second case the sector of fall as before.

In the mixed sector we choose the base point at infinity in the growth direction, $b = +\infty \cdot a$. More precisely, we consider the ray $R_z = z + \mathbb{R}_+ a = \{\zeta = z + sa \colon s \in \mathbb{R}_+\}$ (with the orientation inherited from $\mathbb{R}_+$) and the integral operator $\mathbf{S}_+ \colon f \mapsto \mathbf{S}_+ f$,

$$\begin{aligned} \mathbf{S}_+ f(z) &= -\int_{R_z} e^{\mu(z-\zeta)} f(\zeta)\, d\zeta \\ &= -a \cdot \int_0^{+\infty} e^{-s\cdot\mu a} f(z+sa)\, ds, \qquad s \in \mathbb{R}_+. \end{aligned} \tag{20.30}$$

This integral converges since both the function $e^{-s\mu a}$ and $f(z+sa)$ decrease very fast as $s \to +\infty$. Note that since the sector S' was assumed acute, we can always delete a bounded subset so that the remaining infinite set is convex. For convex domains the construction is always well defined.

In the sector of fall we choose the base point $b = r$ on the "exterior circumference" of the sector S', and fix the initial condition $y(b) = 0$. Then the solution $y(\cdot)$ is given by the integral operator $\mathbf{S}_-$ along the segment $[r, z] = -[z, r] = \{z - sa \colon 0 \leqslant s \leqslant |z-r|\}$, where $a = a(z) = (z-r)/|z-r|$,

$$\begin{aligned} \mathbf{S}_- f(z) &= -\int_{[z,r]} e^{\mu(z-\zeta)} f(\zeta)\, d\zeta \\ &= -a \cdot \int_0^{|z-r|} e^{s\cdot\mu a} f(z-sa)\, ds, \quad a(z) = \frac{z-r}{|z-r|}. \end{aligned} \tag{20.31}$$

There is no question of convergence, since the segment is always finite.

Definition 20.25. Given the sector S' and a nonzero complex number μ such that $\operatorname{Re} \mu z \neq 0$ on the boundary of S', we denote by $\mathbf{S} = \mathbf{S}_{\mu,S'}$ the appropriate integral operator,

$$\mathbf{S}_{\mu,S'} = \begin{cases} \mathbf{S}_+, & \text{if } \operatorname{Re}\mu a > 0 \text{ for some } a \in S', \\ \mathbf{S}_-, & \text{if } \operatorname{Re}\mu z/|z| \leqslant \delta_0 < 0 \text{ for all } z \in S'. \end{cases} \tag{20.32}$$

Denote $\mathcal{O}(S'; N)$ the space of functions holomorphic in the sector S' and decreasing as fast as $O(|z|^{-N})$ for a nonnegative number $N \geqslant 0$. This space can be equipped with the weighted sup-like norm

$$\|f\|_N = \|f\|_{S';N} = \sup_{z \in S'} |z|^N \, |f(z)|. \tag{20.33}$$

Lemma 20.26. *The operator $\mathbf{S}_{\mu,S'}$ is bounded as a linear operator acting on the subspace $\mathcal{O}(S'; 0)$.*

Moreover, it remains bounded when considered as an operator on the space $\mathcal{O}(S'; N)$.

Proof. We fix the sector S' and treat separately the two possibilities of S' being mixed sector or fall sector, depending on the choice of μ. First we consider the case $N = 0$ corresponding to the usual sup-norm.

If S' is the mixed sector and $\|f\| = 1$, that is, $|f(z)| \leqslant 1$, then $|\mathbf{S}_+ f(z)| \leqslant |a| \int_0^\infty e^{-cs} \, ds = |a|/c$, $c = \operatorname{Re}\mu a > 0$.

If S' is the sector of fall, then $|\mathbf{S}_- f(z)| \leqslant |a| \int_0^{|z-r|} e^{cs} \, ds \leqslant 1/|c|$, where $c = c(z) = \operatorname{Re}\mu a(z)$. If z belongs to the translate $r + S'$ of the sector S', then $a(z) = (z - r)/|z - r|$ of modulus 1 belongs to S', hence by the second assumption (20.32) we have $|c(z)| \geqslant \delta_0 > 0$ bounded from below. This proves that $\mathbf{S}_- f$ is bounded in $r + S'$.

Moreover, one can replace S' by another sector $S'' \supset S'$ of slightly bigger opening but still a fall sector; the above arguments would prove then that $\mathbf{S}_- f$ is bounded in $r + S''$. It remains to notice that the difference $S' \setminus (r + S'')$ is bounded, its diameter depending only on S', S'' and r, so the integral (20.31) is bounded there as well. Thus we have proved the boundedness of $\mathbf{S}_-$ with respect to the usual sup-norm $\| \cdot \|_0$ on S'.

To prove the boundedness with respect to the "weighted sup-norms" $\| \cdot \|_N$, assume that $\|f\|_N \leqslant 1$, i.e., $|f(z)| \leqslant |z|^{-N}$, and consider again both possibilities for S'.

Let S' be a mixed sector. Since S' is acute and $z, a \in S'$, we have $|z + sa| \geqslant c' \, |z|$ for some constant $c' > 0$ depending only on S' and all $s \in \mathbb{R}_+$, by obvious geometric considerations. Substituting this inequality into the integral (20.30), we majorize $\mathbf{S}_+ f$ in S' by $|c'z|^{-N} \cdot /|c|$. This proves the boundedness of $\mathbf{S}_+$.

To see why $\mathbf{S}_-$ is bounded in $r + S''$ with respect to this norm (where S'' is chosen as in the case $N = 0$), we split the segment of integration $[r, z]$

in (20.31) into two equal parts. On the initial part $\zeta \in [r, \frac{1}{2}(r+z)]$ the exponential factor $e^{\mu(z-\zeta)}$ is exponentially small, since $|z-\zeta| \geqslant \frac{1}{2}|z|$. On the distant part $\zeta \in [\frac{1}{2}(z+r), z]$ we have the inequality $|\zeta| \geqslant \frac{1}{2}|z|$ and hence by our assumption on f, $|f(\zeta)| \leqslant 2^{-N}|z|^{-N}$, so that the full integral $\mathbf{S}_- f(z)$ is bounded by $2^{-N}|z|^{-N}/|c(z)|$. Exactly as in the case $N=0$, this implies that $\mathbf{S}_-$ is bounded in the $\|\cdot\|_N$-norm. □

Remark 20.27. In all these constructions the bound for the norm $\|\mathbf{S}_\pm\|_{S';N}$ may depend on N and the opening of the sector S' but does not depend on the "size" (the parameter r) of the sector. This can be verified independently by the rescaling arguments.

20L. Integral equation and demonstration of Theorem 20.23. If instead of the simple equation (20.28) we would have a slightly more general form

$$\tfrac{d}{dz}y = [\mu + G(z)]y + g(z), \tag{20.34}$$

then the method of variation of constants, instead of giving an explicit solution, would reduce (20.34) to an integral equation.

After the substitution $y(z) = e^{\mu z}y'(z)$ (20.34) is transformed to the equation $\frac{d}{dz}y'(z) = e^{-\mu z}[G(z)y(z) + g(z)]$, which after taking primitive and multiplication by $e^{\mu z}$ yields

$$y(z) = e^{\mu z}y(b) + \int_b^z e^{\mu(z-\zeta)}[G(z)y(z) + g(z)]\,dz.$$

Again the base point b can be chosen freely, and this freedom can be again used to ensure the flatness of solutions. As before, we conclude that

$$y = \mathbf{S}[Gy + g], \qquad \mathbf{S} = \mathbf{S}_{\mu,S'}, \tag{20.35}$$

if it exists, satisfies the differential equation (20.34).

A multidimensional generalization of this example for the k-dimensional system (20.27) is straightforward. Denote by $\mathbf{S}$ the diagonal integral operator defined on vector functions bounded in the sector S', as follows:

$$\mathbf{S}(y_1, \dots, y_k) = (\mathbf{S}_1 y_1, \dots, \mathbf{S}_k y_k), \qquad \mathbf{S}_i = \mathbf{S}_{\mu_i,S'},\ i = 1, \dots, k. \tag{20.36}$$

This operator, a Cartesian product of integral operators of the form (20.32), depends on the eigenvalues of the diagonal matrix $D = \operatorname{diag}\{\mu_1, \dots, \mu_k\}$, with the path of integration being in general different for each component.

In complete analogy with (20.35), solution of the system (20.27) can be constructed by solving the integral equation

$$y = \mathbf{S}[Gy + g], \qquad \mathbf{S} = \operatorname{diag}\{\mathbf{S}_1, \dots, \mathbf{S}_k\}. \tag{20.37}$$

The diagonal integral operator $\mathbf{S}$ is bounded by Lemma 20.26, if the boundary rays of S' are not exceptional for any μ_i, that is, not separation rays for the initial system (20.5). We show that the composition occurring in the right hand side of (20.37) is a contraction, if the sector $S' = \{|z| > r,\ |\operatorname{Arg} z| < \pi - \delta\}$ is sufficiently small, i.e., r is sufficiently large.

Proposition 20.28. *In the assumptions of Theorem 20.23 the operator*

$$y \mapsto \mathbf{G}y = Gy + g$$

is Lipschitz in the sense of any norm $\|\cdot\|_{S';N}$ *on the space of vector functions holomorphic in* $S'_r = S' \cap \{|z| > r\}$,

$$\|\mathbf{G}y - \mathbf{G}y'\|_{S'_r;N} < \rho\,\|y - y'\|_{S'_r;N}, \qquad \rho = \rho(r) > 0.$$

The Lipschitz constant $\rho(r)$ *tends to zero as* $r \to +\infty$.

Proof. The Lipschitz constant $\rho = \rho(r)$, actually independent of N, can be chosen as $\rho(r) = \sup_z\{|G(z)|\colon\ z \in S'_r\}$. Indeed,

$$\|\mathbf{G}y - \mathbf{G}y'\|_{S'_r;N} \leqslant \sup_{z \in S'_r} |z|^{-N}|G(z)| \cdot |y(z) - y'(z)| \leqslant \sup_{z \in S'_r} |G(z)| \cdot \|y - y'\|_{S'_r,N}.$$

By assumption, $G(z)$ tends to zero as $z \to \infty$ in S', hence $\rho(r) \to 0^+$ as $r \to +\infty$. □

Proof of Theorem 20.23. Our goal already has been reduced to showing that the integral equation (20.37) admits a solution flat in the sector S'. Without loss of generality we may assume that the rays bounding S' are not exceptional (otherwise one can increase slightly the opening while keeping the sector acute).

Let $N \geqslant 0$ be an arbitrary order of decay. As soon as r is sufficiently large, $r \geqslant r(N)$, the Lipschitz constant $\rho(r)$ of the operator $\mathbf{G}$ becomes smaller than the bound for the norm of the operator $\mathbf{S}$ with respect to any given N (recall that $\|\mathbf{S}\|_N$ does not depend on r; see Remark 20.27). In the corresponding $S'_r = S' \cap \{|z| > r(N)\}$ the composition $\mathbf{S} \cdot \mathbf{G}$ will be contracting in the $\|\cdot\|_N$-norm. Hence the fixed point-type integral equation (20.37) possesses a *unique* solution, a vector function with each component belonging to the space $\mathcal{O}(S'_N, N)$. Any such solution can in fact be extended to a function holomorphic in the entire sector S' by virtue of the differential equation (20.27) nonsingular in S'. By the uniqueness, any two such extensions necessarily coincide with each other on the intersection of their domains. Together they yield a vector function $y(z)$ holomorphic in S' and decreasing faster than $|z|^{-N}$ for any N as $|z| \to \infty$. In other words, the constructed solution $y(z)$ is flat as required. □

20M. Sector enlargement and the proof of Sibuya Theorem 20.16. Let S be an "acute" sector of opening $\pi/(m-1)-2\delta$ as in (20.22). Consider its rotations $S_\pm = e^{\pm 2i\delta}S$: the union of the three sectors $S \cup S_+ \cup S_-$ is a sector of opening $\pi/(m-1)+2\delta$. By assumption, each sector $S_\pm$ may contain only those separation rays, that already were contained in S (and perhaps not all of them).

Since $S, S_\pm$ are all "acute", by Corollary 20.24 there exist normalizing cochains $H, H_\pm$ conjugating the initial system with its formal normal form. Therefore for suitable Stokes matrices $C_\pm$ (not to be confused with the Stokes collection of the initial system),

$$H(t) = H_\pm(t)WC_\pm W^{-1}(t) \quad \text{on the intersections } S_\pm \cap S, \qquad (20.38)$$

where $W(t)$ is a fixed diagonal solution of the formal normal form. But since the flaps $S_\pm \smallsetminus S$ contain no separation rays, the difference $E - W(t)C_\pm W^{-1}(t)$ remains flat not only on the intersections $S_\pm \cap S$, but also on the sectors $S_\pm$. In other words, the right hand side of (20.38) extends the same series $\widehat{H}$ and provides an analytic continuation of H on the larger sector $S \cup S_\pm$.

Exercises and Problems for §20.

Problem 20.1. Let $J \in \operatorname{Mat}(n, \mathbb{C})$ be an upper-triangular standard nilpotent Jordan block of maximal size, and ad_J the linear operator of commutation with J. Prove that the linear subspace of matrices having zeros in all places except for the last row, is transversal (complementary) to the image of ad_J.

Problem 20.2 (Demonstration of the Borel–Ritt theorem after [**Was87**]). Let $\varphi(c, \beta; t) = 1 - \exp(-ct^{-\beta})$, $0 < \beta < 1$, $c > 0$, be a function holomorphic in a sector S of opening less than 2π. For an arbitrary formal series $\widehat{F} = \sum_{k=1}^{\infty} a_k t^k$ consider the series $F = \sum_{a_k \neq 0} a_k \varphi(|a_k|^{-1}, \beta; t)\, t^k$.

(a) Prove that $|1 - \exp z| < |z|$ if $\operatorname{Re} z < 0$. (b) Prove that for some $\beta \in (0,1)$ depending on S, the function $-t^{-\beta}$ has negative real part in S. (c) Prove that the series F is majorized by the series $\sum_{a_k \neq 0} |t|^{k-\beta}$ in the sector S. (d) Prove that the series F uniformly converges in S. (e) Prove that the asymptotic series for F coincides with $\widehat{F}$. (f) Prove the Borel–Ritt theorem.