Exercise 18.3. Prove that the Riemann–Hilbert problem can be always solved by a Fuchsian linear system for any monodromy data if the meromorphic matrix form is allowed to have a single extra singular point with identical holonomy at any preassigned point off the singular locus Σ.

Problem 18.4. Construct an example of an irregular singularity and a subspace invariant by the (local) monodromy, which does not extend as an invariant holomorphic subbundle over a neighborhood of the singular point (cf. with Proposition 18.8).

Problem 18.5. Prove that any meromorphic rectangular matrix function $X(t)$ of size $n \times k$, $k < n$ can be locally near $t \in (\mathbb{C}, 0)$ represented under the form $X(t) = L(t)D(t)R(t)$, where $L(t)$ and $R(t)$ are holomorphic invertible square matrices of sizes $n \times n$ and $k \times k$ respectively, and $D(t)$ is the rectangular truncation (first k columns) of a diagonal matrix which has only integer powers t^{ν_i} or zeros on the diagonal.

Exercise 18.6. Prove that any operator $M \in \mathrm{GL}(n, \mathbb{C})$ has at least one invariant subspace $L_k \subset \mathbb{C}^n$ of each intermediate dimension $k = 1, \dots, n-1$.

Problem 18.7. Prove that any two matrix logarithms A, A' of the same monoblock operator differ by an integer multiple of the identity matrix modulo conjugacy:

$$\exp A = \exp A' \text{ is a monoblock } \implies A - CA'C^{-1} = 2\pi i k E$$

for a suitable integer number $k \in \mathbb{Z}$ and an invertible conjugacy matrix $C \in \mathrm{GL}(n, \mathbb{C})$. Prove that each logarithm is also a monoblock.

Problem 18.8. Prove that the Riemann–Hilbert problem is always solvable in the classical sense (i.e., on the trivial bundle) in dimension 2.

Problem 18.9. Prove that the monodromy data with one diagonal matrix can be realized by infinitely many nonequivalent Fuchsian systems.

Problem 18.10. Prove that any irreducible monodromy data can be realized by infinitely many nonequivalent Fuchsian systems.

Problem 18.11. Prove that the Riemann–Hilbert problem is nonsolvable in all dimensions greater than 4.

Problem 18.12. Prove the following generalization of Theorem 18.12. Let ∇ be a meromorphic *non-Fuchsian* connexion on a holomorphic vector bundle of rank n and the splitting type $D = \{d_1, \dots, d_n\}$ with at least one Fuchsian singularity. Denote by m the total order of poles of all singularities. Prove that if for some pair of indices $|d_i - d_j| \geqslant (m-2)(n-1)$, then the connexion ∇ is reducible, i.e., has an invariant subbundle.

19. Linear nth order differential equations

Linear high order scalar differential equation can be reduced to a rather special class of *companion* linear systems which are naturally defined connexions on the *jet bundle*. Because of the special form, regular singular points of such connexions can be easily identified and explicit meromorphic transformation bringing them to the Fuchsian form is well known since L. Fuchs himself. However, this meromorphic transformation is nontrivial and globally Fuchsian equations on the Riemann sphere $\mathbb{P}$ naturally “live” on nontrivial holomorphic vector bundles, whose type depends on the number of singular points.

An additional feature, an important tool of investigation, is the structure of (noncommutative) algebra on the set of linear differential operators, which implies the possibility of *factorization* of operators. The latter circumstance plays an important role when studying *roots of solutions* of linear ordinary differential equations.

At the end of the section we address several questions in the spirit of the Riemann–Hilbert problem for linear high order equations in the cases where these questions make sense.

19A. High order differential operators: algebraic theory. Let T be a Riemann surface (complex 1-dimensional manifold). Denote by $\mathcal{M} = \mathcal{M}(T)$ the field (commutative $\mathbb{C}$-algebra) of meromorphic functions on T. Any derivation $D \in \operatorname{Der}\mathcal{M}$, a $\mathbb{C}$-linear self-map of $\mathcal{M}$ into itself which satisfies the Leibnitz rule $D(fg) = f\,Dg + G\,Df$, is associated with a meromorphic vector field on T,

$$\operatorname{Der}\mathcal{M} \cong \mathcal{D}(T) \otimes \mathcal{M}.$$

Since T is one-dimensional, any two derivations differ by a meromorphic multiplier,

$$D, D' \in \operatorname{Der}\mathcal{M} \iff D' = rD, \qquad \text{for some} \quad r \in \mathcal{M}. \tag{19.1}$$

Definition 19.1. A *linear nth order differential operator* is any $\mathbb{C}$-linear operator $L\colon \mathcal{M} \to \mathcal{M}$, which admits a representation

$$\begin{aligned} L &= a_0\,D^n + a_1\,D^{n-1} + \cdots + a_{n-1}D + a_n, \\ D &\in \operatorname{Der}\mathcal{M}, \quad a_0, a_1, \dots, a_n \in \mathcal{M}, \quad a_0 \not\equiv 0. \end{aligned} \tag{19.2}$$

The operator $a_0 D^n$ is called the *leading term* of L. The operator L is called *monic* (more precisely, D-monic), if $a_0 = 1$. A linear nth order homogeneous differential equation is the equation of the form

$$Lf = 0. \tag{19.3}$$

This definition formally depends on the choice of the derivation D, yet one can immediately verify using (19.1) and the Leibnitz rule, that an expansion (19.2) can be re-expanded (with different coefficients, but of the same degree) in powers of any other derivation D'. We will denote

$$\mathfrak{D}(n,T) = \{L\colon \mathcal{M}(T) \to \mathcal{M}(T),\ \operatorname{ord} L = n\},$$
$$\mathfrak{D}(T) = \bigcup_{n\geqslant 0} \mathfrak{D}(n,T).$$

Differential operators of order 0 are multiplications by scalar functions and hence can be identified with the algebra $\mathcal{M} = \mathcal{M}(T)$ itself. The collection of differential operators of all orders is naturally filtered by the order.

The space of all differential operators $\mathfrak{D}(T)$ forms a noncommutative associative algebra by composition:

$$L, L' \in \mathfrak{D}(T) \implies LL', L'L \in \mathfrak{D}(T),$$
$$\operatorname{ord} LL' = \operatorname{ord} L'L = \operatorname{ord} L + \operatorname{ord} L'.$$

The only units of $\mathfrak{D}(T)$ are zero order operators corresponding to multiplication by a nonzero meromorphic function[5]. Though the algebra $\mathfrak{D}(T)$ is noncommutative, it has many features similar to that of the commutative algebra $\mathcal{M}[D]$ of polynomials in a single indeterminate D with coefficients in the ring $\mathcal{M} = \mathcal{M}(T)$ of meromorphic functions. Thus, the representation (19.2) can be considered now as a (noncommutative) polynomial expansion in $\mathfrak{D}(T)$ in powers of the derivation $D \in \operatorname{Der}\mathcal{M}(T)$ with all coefficients occurring *to the left* of all powers $D, D^2, \dots, D^n$. Another feature is the possibility of division with remainder similar to the division of univariate polynomials.

Lemma 19.2. *For any two operators $L \in \mathfrak{D}(n,T)$ and $Q \in \mathfrak{D}(k,T)$ of orders $n \geqslant k$, then there exist two operators P (the incomplete ratio) and R (the remainder), such that*

$$L = PQ + R, \qquad \operatorname{ord} P = \operatorname{ord} L - \operatorname{ord} Q, \quad \operatorname{ord} R < \operatorname{ord} Q. \tag{19.4}$$

Proof. The operators P, R can be constructed by the following algorithm which is a modification of the division algorithm for polynomials in one variable. If the operators L, Q are expanded in powers of any derivation $D \in \operatorname{Der}\mathcal{M}$ as follows:

$$\begin{aligned} L &= a_0 D^n + a_1 D^{n-1} + \cdots + a_n, \\ Q &= b_0 D^k + b_1 D^{k-1} + \cdots + b_k, \end{aligned} \qquad a_i, b_j \in \mathcal{M}, \tag{19.5}$$

then the leading term of the operator $D^{n-k}Q$ is $b_0 D^n$ and hence the operator $L_1 = L - P_0 Q$, where $P_0 = (a_0/b_0)D^{n-k}$, has the order $\leqslant n-1$. Repeating this step, we construct P_1 so that $L_2 = L_1 - P_1 Q$ is of the order strictly inferior to that of L_1, *etc.*

[5]The property of linear operators on the algebra $\mathcal{M}$ to be *differential operators* can be defined in purely algebraic terms of commutation with the units of the algebra of self-maps (Problem 19.1).

After at most $n-k$ steps we will be left with an operator of order strictly less than k, which is designated to be the residue R. The "partial incomplete ratios" $P_0, P_1, \dots$ add together to form the operator $P = P_0 + P_1 + \cdots$. $\square$

Remark 19.3. Assume that all coefficients a_i, b_j of the operators L and Q in (19.5) are holomorphic at a given point $t_0 \in T$, and the leading coefficient b_0 of the divisor Q is nonvanishing at this point, $b_0(t_0) \neq 0$. Under these assumptions both the remainder and the incomplete ratio will be obtained as expansions in powers of D with coefficients holomorphic at t_0. This can be seen by direct inspection of the algorithm.

Definition 19.4. An operator $L \in \mathfrak{LO}(n,T)$ is *divisible* by $Q \in \mathfrak{LO}(k,T)$, if $L = PQ$ with $P \in \mathfrak{LO}(n-k,T)$. An operator L is *reducible*, if it is divisible by an operator $Q \in \mathfrak{LO}(k,T)$ with $0 < k < n$. Otherwise L is called *irreducible*.

19B. Linear ordinary differential equations: the naïve approach. Linear high order equations can be considered as a particular case of linear systems of first order differential equations of a special, so-called *companion form*.

In this section it will be convenient to enumerate coordinates of the complex space $\mathbb{C}^{n+1} = \{(x_0, \dots, x_n)\}$ starting from the zero index value, x_0. Let $D \in \operatorname{Der}\mathcal{M}$ be an arbitrary derivation; for instance, if $T = \mathbb{C}$ or $T = \mathbb{P}$, we may assume that $D = \frac{\partial}{\partial t}$. Then, denoting the unknown function by $y = x_0$ and its derivatives by $x_k = D^k y$, $k = 1, \dots, n$, we reduce the scalar equation (19.3) to the system

$$Dx = A(t)x, \quad \text{where:} \qquad D \in \operatorname{Der}\mathcal{M}, \quad A \in \operatorname{Mat}(n, \mathcal{M}), \qquad A = \begin{pmatrix} 0 & 1 & & & \\ & 0 & 1 & & \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ & & & 0 & 1 \\ -\frac{a_n}{a_0} & -\frac{a_{n-1}}{a_0} & \cdots & -\frac{a_2}{a_0} & -\frac{a_1}{a_0} \end{pmatrix}, \tag{19.6}$$

called the *companion system* for the linear equation (19.2)–(19.3). This reduction immediately allows us to reformulate for linear equations all results from §15.

Definition 19.5. A *regular* (*nonsingular*) *point* of a linear equation (19.2)–(19.3) is any point $t_0 \in T$ at which the vector field $D \in \mathcal{D}(T)$, associated with the derivation $D \in \operatorname{Der}\mathcal{M}(T)$, is nonsingular and all ratios $a_i(t)/a_0(t)$ are holomorphic (have no poles).

Nonregular points are naturally called *singularities* of the equation and denoted by $\operatorname{Sing} L$. A singular point is called *regular*, if it is regular for the companion system in the sense of Definition 16.2.

In other words, a singular point for the equation is regular, if all solutions of the equation together with their derivatives grow moderately (in the sense of Definition 16.1) as t tends to t_0.

Proposition 19.6. *Solutions of the linear equation* (19.2)–(19.3) *locally exist near any nonsingular point and admit unique analytic continuation along any path free from singularities of this equation.*

Dimension (*over* $\mathbb{C}$) *of the space of solutions of this equation in any simply connected domain free from singularities of the equation, is equal to the order of the equation.*

Proof. The first assertion is a reformulation of Theorem 15.3 for the companion system.

The second assertion immediately follow from the fact that the linear map which assigns to every holomorphic function $f(\cdot)$ the initial conditions,

$$f(\cdot) \longmapsto \big(f(t_0), \tfrac{d}{dt}f(t_0), \dots \tfrac{d^{n-1}}{dt^{n-1}}f(t_0)\big) \in \mathbb{C}^n,$$

becomes a linear isomorphism between *solutions* of the linear equation (19.2)–(19.3) and the space of initial conditions. Injectivity of this map is the uniqueness part, and surjectivity the uniqueness part of Theorem 15.3. $\square$

Proposition 19.6 implies that solutions of a linear equation $Lf = 0$ are holomorphic functions eventually ramified over the singular locus $\Sigma = \operatorname{Sing} L$. Since analytic continuation along paths preserves the space of solutions of this equation, the operator of analytic continuation Δ_γ along any loop $\gamma \in \pi_1(T \smallsetminus \Sigma, t_0)$ acts by a linear transformation on the row vector of functions,

$$\Delta_\gamma(f_1, \dots, f_n) = (f_1, \dots, f_n) \cdot M_\gamma, \qquad M_\gamma \in \mathrm{GL}(n, \mathbb{C}), \tag{19.7}$$

where M_γ are the *monodromy matrices*. In the future any tuple of holomorphic functions satisfying the monodromy property (19.7), will be called a *monodromic* tuple.

The monodromy property is almost sufficient for a collection of functions to satisfy a linear differential equation with meromorphic coefficients. The additional requirement is regularity of all singular points.

Theorem 19.7 (G. F. B. Riemann)**.** *A monodromic tuple of n functions regular at each ramification point of a finite set $\Sigma \subset T$, satisfies a linear ordinary differential equation $Lf = 0$ with meromorphic coefficients, $L \in \mathfrak{LO}(k, T)$, $k \leqslant n$.*

This equation can be explicitly written using Wronskians; see Proposition 19.9.

Definition 19.8. The *Wronskian*, or Wronski determinant, of n functions is the determinant of the Wronski matrix,

$$W(f_1,\dots,f_n) = \det\begin{pmatrix} f_1 & f_2 & \dots & f_n \\ Df_1 & Df_2 & \dots & Df_n \\ \vdots & \vdots & \ddots & \vdots \\ D^{n-1}f_1 & D^{n-1}f_2 & \dots & D^{n-1}f_n \end{pmatrix}. \tag{19.8}$$

The Wronskian is a holomorphic (resp., meromorphic) function of $t \in U \subset T$ if all functions $f_1,\dots,f_n$ were holomorphic (resp., meromorphic) and D is holomorphic vector field in U.

The Wronskian depends multi-linearly (over $\mathbb{C}$) and anti-symmetrically on the functions f_j. In particular, it vanishes *identically* if the functions f_j are linearly dependent over $\mathbb{C}$. If $f_1,\dots,f_n$ are solutions of a linear equation (19.3), then $W(f_1,\dots,f_n)$ is the determinant of the matrix solution $X(t)$ of the associated companion system (19.6). By the Liouville–Ostrogradskii theorem (Problem 15.10),

$$Dw = -\frac{a_1(t)}{a_0(t)}w, \qquad w = W(f_1,\dots,f_n). \tag{19.9}$$

From this identity it follows that a Wronskian of n solutions of a linear equation is either nonvanishing everywhere outside the singular locus, or vanishes identically.

The Riemann theorem follows immediately from the following assertion.

Proposition 19.9 (gloss of Riemann Theorem 19.7)**.** *For any regular monodromic tuple $f_1,\dots,f_n$ such that the Wronskian $w(t) = W(f_1,\dots,f_n)(t)$ is not identically zero, the operator*

$$L = w^{-1}W(f_1,\dots,f_n,\,\cdot\,), \qquad Lf = w^{-1}W(f_1,\dots,f_n,f), \tag{19.10}$$

is a monic differential operator of order n with meromorphic coefficients,

$$L = D^n + a_1D^{n-1} + \cdots + a_{n-1}D + a_n \in \mathfrak{D}(n,T), \qquad a_i \in \mathcal{M}, \tag{19.11}$$

vanishing on all functions $f_1,\dots,f_n$.

Proof. To prove that L is a monic differential operator, we expand the "large" $(n+1)\times(n+1)$-determinant $W(f_1,\dots,f_n,f)$ in the elements of the last column containing the derivatives of f. The coefficients a_i of the expansion are $n\times n$-minors of the "large" matrix, formed by the first n columns. The leading coefficient (before the highest derivative) is exactly the minor $w = W(f_1,\dots,f_n)$. After division by w we conclude that L is a monic differential operator with the coefficients which are ratios of the minors.

All these minors have the same monodromy (the corresponding matrices are multiplied from the right by the same matrix factors M_γ), hence the ratios of their determinants are single-valued. Because of the regularity, the singularities of these ratios are finite order poles.

Since the Wronskian vanishes when any two columns coincide, each f_j belongs to the null space of L. □

Remark 19.10 (warning). The singular locus of the operator (19.11) can be larger than the ramification locus of the monodromic tuple $(f_1, \dots, f_n)$.

19C. Factorization of differential operators. Solutions of a linear differential equation in general do not belong to the field $\mathfrak{M} = \mathfrak{M}(T)$, but rather to some bigger field (extension) $\mathfrak{M}' \supseteq \mathfrak{M}$. This field can be obtained by formally adjoining these solutions and their derivatives of order $< n$. The extension field, denoted by

$$\mathfrak{M}' = \mathfrak{M}(f_1, \dots, f_n) = \mathfrak{M}(L),$$

is called the *Picard–Vessiot extension* of the initial field $\mathfrak{M} = \mathfrak{M}(T)$.

Picard–Vessiot extensions are *differential fields* (i.e., any derivation $D \in \operatorname{Der} \mathfrak{M}$ extends as a derivation to $\operatorname{Der} \mathfrak{M}'$) with the same subfield of constants (i.e., $Du = 0$, $u \in \mathfrak{M}'$, is possible if and only if $u = \text{const} \in \mathbb{C}$). Besides formally algebraic construction of such extensions, they can be identified with subfields of the field $\mathfrak{M}(T, t_0)$ of meromorphic germs at a nonsingular point $t_0 \notin T$.

In the same way as any polynomial admits factorization by linear terms over the field obtained by adjoining its roots to the field of the coefficients, every linear differential operator can be represented as a composition of first order operators with coefficients in $\mathfrak{M}' = \mathfrak{M}(L)$.

We start with an observation that divisibility of operators can be easily described in terms of common solutions.

Proposition 19.11. *An operator $L \in \mathfrak{D}(T)$ is divisible by another operator $Q \in \mathfrak{D}(T)$, if and only if any solution of $Qf = 0$ is also a solution of $Lf = 0$.*

Proof. The "if" part is obvious. To prove divisibility, consider a fundamental system $f_1, \dots, f_k$ of solutions of the equation $Qf = 0$ and divide L by Q with remainder R, $L = PQ + R$, as in Lemma 19.2. Being in the null space for L and Q by assumption, $f_1, \dots, f_k$ also belong to the null space of PQ and hence to the null space of R. Since $\operatorname{ord} R < k$, this is possible only when $R = 0$ by Proposition 19.6. □

For any meromorphic germ $0 \not\equiv f \in \mathfrak{M}(T, t_0)$ one can immediately construct a first order linear operator vanishing on this germ, e.g., in the form

$$Q = fD - f', \qquad f' = Df.$$

By Proposition 19.6, any operator L such that $Lf = 0$, can be divided by Q, $L = L'Q$. If another solution (germ) $g \in \mathcal{M}(T, t_0)$, is known, $Lg = 0$, then the germ $g' = Qg$ is a meromorphic solution of the equation $L'g' = 0$ and can be used to further factor the operator L'.

If all n solutions $f_1, \dots, f_n$ of the homogeneous nth order equation $Lf = 0$ are known, this procedure allows us to construct *complete factorization* of L as a composition of n first order operators with coefficients in $\mathcal{M}' = \mathcal{M}(f_1, \dots, f_n)$. The factorization involves Wronskians, or Wronski determinants of the functions.

Now we can describe the factorization of an arbitrary differential operator $L \in \mathfrak{D}(T)$ with a known system of n linearly independent solutions $f_1, \dots, f_n$, using the Wronskians of these functions. Assume that U is a simply connected domain without singularities of L, so that $f_1, \dots, f_n \in \mathcal{O}(U)$, and denote by

$$\begin{gathered} w_k = W(f_1, \dots, f_k) \in \mathcal{O}(U), \qquad k = 1, \dots, n, \\ w_{-1} = w_0 = 1, \qquad w_{n+1} = w_n, \end{gathered} \tag{19.12}$$

the Wronskians of the first k functions from the ordered tuple $f_1, \dots, f_n$ (the functions w_{-1}, w_0 and w_{n+1} are introduced for convenience).

Theorem 19.12. *If $f_1, \dots, f_n \in \mathcal{O}(U)$ are linearly independent solutions of the equation $Lf = 0$ with a monic operator $L = D^n + \cdots$, then L is a composition of n derivations D interspersed with $n + 1$ multiplications $b_0, \dots, b_n \in \mathcal{M}(U) \cong \mathfrak{D}(0, U)$, as follows:*

$$\begin{gathered} L = b_n\, D\, b_{n-1}\, D\, b_{n-2} \cdots b_2\, D\, b_1\, D\, b_0, \\ b_k = \frac{w_k^2}{w_{k-1} w_{k+1}}, \quad k = 0, 1, \dots, n. \end{gathered} \tag{19.13}$$

Proof. Consider the monic differential operators L_k of order $k = 0, 1, \dots, n$,

$$L_0 = \mathrm{id}, \quad L_k = w_k^{-1}(t) \cdot W(f_1, \dots, f_k, \cdot), \qquad k = 1, \dots, n.$$

We claim that these operators satisfy the operator identity

$$D\, \frac{w_{k-1}}{w_k}\, L_{k-1} = \frac{w_{k-1}}{w_k}\, L_k, \qquad k = 1, \dots, n. \tag{19.14}$$

Indeed, both parts are differential operators of the same order k with the same leading terms $(w_{k-1}/w_k)\, D^k$. The null spaces of both operators also coincide with the linear span of $f_1, \dots, f_k$ and hence with each other. Indeed, the functions $f_1, \dots, f_{k-1}$ obviously belong to the null space of both parts. On the last function f_k the operator L_k vanishes by definition, whereas $L_{k-1} f_k = w_k / w_{k-1}$, so the left hand side of (19.14) also vanishes. Being both monic and having the same null space, the operators occurring in the two sides of (19.14), must coincide.

The identity (19.14) can be rewritten as

$$L_k = \frac{w_k}{w_{k-1}}\, D\, \frac{w_{k-1}}{w_k} L_{k-1}, \qquad k = 1, \dots, n.$$

Applying it recursively to the monic operator $L = L_n$ which is what we are interested in by Proposition 19.9, we obtain its decomposition into n terms

$$L_n = \left(\frac{w_n}{w_{n-1}}\, D\, \frac{w_{n-1}}{w_n}\right) \cdots \left(\frac{w_2}{w_1}\, D\, \frac{w_1}{w_2}\right) \cdot \left(\frac{w_1}{w_0}\, D\, \frac{w_0}{w_1}\right) \cdot L_0,$$

which coincides with (19.13). □

The advantage of such "complete factorization" becomes clear when solving homogeneous or nonhomogeneous equations. Denote by D^{-1} *any* "primitive" operator, i.e., $D^{-1}f = \int f\,dt$ in the case $D = \frac{\partial}{\partial t}$ (defined modulo a constant). Then the general solution of the equation $Lf = g$ for L factored as in (19.13), is given by the symbolic formula

$$f = b_0^{-1}\, D^{-1}\, b_1^{-1}\, D^{-1} \cdots D^{-1}\, b_{n-1}^{-1}\, D^{-1}\, b_n^{-1} g. \tag{19.15}$$

In other words, *knowing a fundamental system of solutions of a homogeneous differential equation allows us to solve any nonhomogeneous equation by taking n quadratures.* This may be a convenient alternative to reducing the equation to the companion system and using the method of variation of constants.

In general, solutions of linear equations, are ramified at singular points hence the formal factorization (19.13) has in general multivalued coefficients. In other words, factorization (19.13) holds over the extension $\mathfrak{M}' \supsetneq \mathfrak{M}$ and *not* over the initial field $\mathfrak{M} = \mathfrak{M}(T)$. Reducibility of operators in over $\mathfrak{M}$ is closely related to reducibility of their monodromy group.

Theorem 19.13. *A linear operator $L \in \mathfrak{LO}(T)$ having only regular singularities in T, is reducible in the algebra $\mathfrak{LO}(T)$ if and only if its monodromy group is reducible (i.e., has a nontrivial invariant subspace).*

Proof. Assume that $L = PQ$ and $f_1, \dots, f_k$ is a fundamental system of solutions for $Qf = 0$. Then these functions also solve the equation $Lf = 0$ and span an invariant subspace of the monodromy group which is therefore reducible. Conversely, assume (without loss of generality) that an invariant subspace of the monodromy group for $Lf = 0$ is generated by the first k functions $f_1, \dots, f_n$ of some fundamental system of solutions. Then by the Riemann Theorem 19.7, there exists an operator $Q \in \mathfrak{LO}(T)$ of order k, annulled by these first functions. By Proposition 19.11, L is divisible by Q and hence reducible in $\mathfrak{LO}(T)$. □

Factorization of operators is compatible with regularity. For brevity we say that a differential operator $L \in \mathfrak{LO}(T)$ is *regular* in $U \subset T$, if it has only regular singular points there.

Lemma 19.14. *Composition of two regular operators is regular. Conversely, if a regular operator is reducible in $\mathfrak{LO}(T)$, then both factors are also regular.*

Proof. If $L = PQ$, then any solution of the equation $Lf = 0$ is a solution of the nonhomogeneous equation $Qf = g$, where g is some solution of the lower order equation $Pg = 0$. For any singular point $t_0 \in T$, the function g grows moderately at t_0 since P is regular. Since Q is also regular at this point, by Lemma 16.6 we conclude that f also grows moderately at t_0. This proves regularity of PQ.

Conversely, if $L = PQ$ is regular, then any function from the null space of Q grows moderately at any singular point t_0 regardless of regularity of P. To prove regularity of P, choose any solution g of the equation $Pg = 0$. As before, let f be any solution of $Qf = g$: by construction, f grows moderately as a solution of $Lf = 0$ and can be represented as

$$f(t) = (h_1, \dots, h_n)\,(t - t_0)^A (c_1, \dots, c_n)^\top,$$

where the row vector function $(h_1, \dots, h_n)$ is meromorphic at t_0, the column vector $(c_1, \dots, c_n)^\top$ has constant entries and A is any logarithm of the monodromy matrix around t_0. Any such function admits any number of derivations and multiplications by meromorphic functions while retaining the moderate growth at t_0. Therefore application of any operator $Q \in \mathfrak{LO}(T)$ proves that $g = Qf$ grows moderately at t_0, so that P is regular. □

As an immediate application of this result, we have the local theorem on complete factorization.

Theorem 19.15. *Any differential operator $L \in \mathfrak{LO}(T)$ having a regular singularity at a point $t_0 \in T$, admits complete factorization in a small neighborhood $U = (T, t_0)$ of this point,*

$$L = P_n P_{n-1} \cdots P_1, \qquad P_i \in \mathfrak{LO}(U), \quad \operatorname{ord} P_i = 1, \tag{19.16}$$

with first order factors P_i having meromorphic coefficients in U and regular singularity at t_0.

Proof. The monodromy group of any operator in a punctured neighborhood U of an (isolated) singular point is cyclic and hence always admits a one-dimensional invariant subspace. By Theorem 19.13, $L = L_0$ is divisible from the right by a first order operator $P_1 \in \mathfrak{LO}(U)$ whose leading term can be prescribed arbitrary. By Lemma 19.14, both P_1 and its left cofactor L_1 are regular at t_0. Thus the process can be continued by induction until the complete factorization is achieved. $\square$

Remark 19.16. Note that the leading terms of $P_1, \dots, P_{n-1}$ can be prescribed arbitrarily, as multiplication by a meromorphic germ is a unit of the algebra $\mathfrak{LD}(n, T)$.

19D. Fuchsian singularities of nth order equation. Similarly to the general case of linear systems, regular singularity is not necessarily a first order pole of the companion system if the derivation D itself is nonsingular at this point. However, unlike the general case, we can introduce the class of equations with "*first order pole*", which turns out to coincide with the class of regular equations.

The reason why the words above are enclosed by the quotation marks, is noninvariance of this notion. Indeed, the companion system (19.6) by definition has a singularity at a point $t_0 \in T$ if either the vector field D is singular at t_0, i.e., $D = r(t)\frac{\partial}{\partial t}$ in a local chart on T with $\operatorname{ord}_{t_0} r(t) > 0$, or D is nonsingular, $\operatorname{ord}_{t_0} r(t) = 0$, but some of the ratios a_i/a_0, $i = 1, \dots, n$ have a pole at t_0 (in such a case we denote by $\operatorname{ord} A$ the negative of the maximal order of the poles of all entries of a meromorphic matrix function $A(t)$). In both cases the order of the pole, understood as $\operatorname{ord} r - \operatorname{ord} A$, is positive. Yet this order *explicitly depends on the choice of the derivation D used to write the companion system.*

Definition 19.17. A differential operator $L \in \mathfrak{LD}(T)$ is *Fuchsian* at a singular point t_0, if in the companion form (19.6)

$$\operatorname{ord}_{t_0} D = 1, \qquad \operatorname{ord}_{t_0} A = 0.$$

This definition is equivalent to another, more transparent (though less invariant) description.

Proposition 19.18. *A differential operator L is Fuchsian at a finite point t_0, if after expansion in the powers of $D' = (t - t_0)\frac{\partial}{\partial t}$ and reduction to the monic form, it has holomorphic coefficients.* $\square$

Obviously, instead of the linear vector field D' one can use any other holomorphic germ with a simple singularity at t_0. Re-expanding an expression for the monic operator $D'^n + \cdots + a_{n-1}D' + a_n$ in powers of the "usual" differentiation $D = \frac{\partial}{\partial t}$, we obtain the property that is often used as the definition of finite Fuchsian singularity [**Inc44, Har82**].

Proposition 19.19. *A monic operator $L = D^n + \cdots + a_n \in \mathfrak{LD}(n, \mathbb{C})$, $D = \frac{\partial}{\partial t}$, has a Fuchsian singularity at a finite point $t = t_0 \in \mathbb{C}$, if and only if* $\operatorname{ord}_{t_0} a_k(t) \geqslant -k$ *for all $k = 0, \dots, n$.* $\square$

The advantage of the invariant Definition 19.17 is that it can automatically be reformulated for the case where the Fuchsian singularity is at infinity, $t_0 = \infty \in \mathbb{P}$ (Problem 19.6).

From the Sauvage Theorem 16.10 we immediately conclude that any Fuchsian singularity of an operator $L \in \mathfrak{D}(T)$ is always regular. Somewhat unusual is the fact that for high order equations the inverse is also true.

Theorem 19.20 (L. Fuchs, 1868)**.** *Any regular singularity of a linear ordinary differential equation with meromorphic coefficients, is Fuchsian.*

Proof. 1°. For equations of the first order the assertion of the theorem is verified by a straightforward computation. Assume that the regular singularity occurs at $t = 0$. Consider the equation $L'f = 0$, where $L = D' + a_1'(t)$ is expanded using the standard Euler derivation $D' = t\frac{\partial}{\partial t}$. If L has a regular singularity at $t = 0$, we can represent its solution as $f(t) = t^\lambda h(t)$ with an appropriate complex $\lambda \in \mathbb{C}$ and some meromorphic function $h(t)$. Changing λ by a suitable integer number, we can assume in addition that h is holomorphic and holomorphically invertible at $t = 0$. Substituting this representation for f into the equation $D'f + a_1'f = 0$, we obtain the formula $-a_1'(t) = D'f/f = \lambda + (D'h/h)$. Since h is holomorphically invertible and $D' = t\frac{d}{dt}$ holomorphic, we conclude that a_1' is holomorphic at t_0 and hence $L = D' + a_1'$ is Fuchsian.

2°. The case of an arbitrary order follows from the factorization Theorem 19.15. By this theorem, any regular operator L can be factored as $L = a_0' P_n \cdots P_1$ with each P_i being a first order operator regular at $t = 0$. Since the leading terms of P_i can be chosen arbitrarily (Remark 19.16), we assume that

$$P_i = tD + a_i' = D' + a_i', \qquad i = 1, \dots, n.$$

By Step 1°, each P_i is Fuchsian, that is, the free terms $a_1', \dots, a_n'$ are necessarily holomorphic at t_0. But then the composition $P_n \cdots P_1$ begins with the leading term D'^n and has all holomorphic coefficients after the complete expansion. In other words, L differs from a Fuchsian operator by a meromorphic factor a_0' and hence is also Fuchsian. □

The companion system can be rewritten in the Pfaffian form. Let $\omega \in \Lambda^1(T) \otimes \mathcal{M}(T)$ be the (scalar) meromorphic 1-form *dual* to the vector field D: by definition, this means that $\omega(D) \equiv 1$. By duality, D has a simple singularity at $t_0 \in T$ if and only if ω has a simple pole at this point. Using this form, the companion system can be written in the Pfaffian form as follows:

$$d\begin{pmatrix} x_0 \\ x_1 \\ \vdots \\ x_{n-2} \\ x_{n-1} \end{pmatrix} = \omega \begin{pmatrix} 0 & 1 & & & \\ & 0 & 1 & & \\ \dots & \dots & \dots & \dots & \dots \\ & & & 0 & 1 \\ b_n & b_{n-1} & \cdots & b_2 & b_1 \end{pmatrix} \begin{pmatrix} x_0 \\ x_1 \\ \vdots \\ x_{n-2} \\ x_{n-1} \end{pmatrix}, \tag{19.17}$$

with holomorphic entries $b_1, \dots, b_n \in \mathcal{O}(T, t_0)$ and a form ω with the first order pole at t_0.

The matrix residue of the corresponding matrix 1-form $\Omega = \omega A$ is equal to $A(t_0) \cdot \operatorname{res}_{t_0} \omega$. Its eigenvalues are called characteristic exponents of the Fuchsian (regular) singularity.

Example 19.21. Any linear ordinary differential equation with a regular singularity at $t = 0$ can be written under the form $Lf = 0$, where

$$L = D'^n + a_1(t)D'^{n-1} + \cdots + a_{n-1}(t)D' + a_n(t), \qquad D' = t\frac{\partial}{\partial t}, \tag{19.18}$$

is the monic expansion in powers of the Euler derivation D' with the coefficients $a_j(t)$ holomorphic at the origin. The characteristic exponents of the corresponding singularity are roots of the polynomial

$$\lambda^n + a_1(0)\lambda^{-1} + \cdots + a_{n-1}(0)\lambda + a_n(0) = 0. \tag{19.19}$$

Obviously, instead of the Euler operator one can use any other operator D'' with a simple singularity and eigenvalue (linearization 1×1-matrix) equal to 1 (see also Problem 19.5).

Fuchsian singularities in the companion form (19.17) are considerably more rigid than general singularities of linear systems, for instance, analytic gauge transform to the Poincaré–Dulac–Levelt normal form destroys the "companion structure". Yet despite all that, one can apply Lemma 16.18 and obtain an ansatz for construction of analytic (ramified) solutions of linear equation near Fuchsian singularity under the form

$$\sum_1^n h_j(t)\, t^{\lambda_j} p_j(\ln t), \qquad h_j \in \mathcal{O}(\mathbb{C}, 0),$$

where $\lambda_1, \dots, \lambda_n$ are characteristic exponents and p_j are polynomials with constant coefficients. The degrees of the polynomials are determined by the resonance identities $\lambda_i \equiv \lambda_j \bmod \mathbb{Z}$ between the characteristic exponents, as encoded by the structure of the matrix I in (16.10).

19E. Jet bundles and invariant constructions. To describe the global structure of regular equations, we need geometric (invariant) description of the jet bundles. We recall briefly their construction; more details can be found in [**AVL91**].

Consider the n-jet space $J^n(T)$ which is the union of all jet spaces at all points of T. The space $J^n(T)$ is equipped with the natural projection $\tau_n\colon J^n(T) \to T$. This projection equips $J^n(T)$ with the structure of a holomorphic vector bundle as follows.

Let $U_\alpha \subset T$ be an open domain and $D_\alpha \in \mathcal{D}(U_\alpha)$ a holomorphic vector field (derivation) *nonsingular in* U_α, as usual identified with the derivation of the algebra $\mathcal{M}(U_\alpha)$. This derivation allows us to associate any jet of a function f at a point p with the (column) vector

$$\text{(jet of } f \text{ at } p \in U_\alpha) \xmapsto{\varPhi_\alpha} (f, Df, D^2 f, \dots, D^n f)^\top \big|_p, \qquad D = D_\alpha. \quad (19.20)$$

The map $\varPhi_\alpha$ defines a trivialization of $J^n(T)$ over the domain U_α.

If U_β is another domain and $D' = D_\beta$ another derivation holomorphic and nonsingular in U_β, then on the intersection $U_{\alpha\beta} = U_\alpha \cap U_\beta$ the two respective derivations $D = D_\alpha$ and $D' = D_\beta = r_{\beta\alpha} D_\alpha$ and their powers are related by the formulas

$$\begin{pmatrix} 1 \\ D' \\ D'^2 \\ \vdots \\ D'^n \end{pmatrix} = \begin{pmatrix} 1 & & & & \\ \vdots & r & & & \\ \vdots & \vdots & r^2 & & \\ \vdots & \vdots & \vdots & \ddots & \\ * & \cdots & \cdots & \cdots & r^n \end{pmatrix} \cdot \begin{pmatrix} 1 \\ D \\ D^2 \\ \vdots \\ D^n \end{pmatrix}, \quad \begin{matrix} D = D_\alpha \in \mathcal{D}(U_\alpha), \\ D' = D_\beta \in \mathcal{D}(U_\beta), \\ r = r_{\beta\alpha} \in \mathcal{O}(U_{\alpha\beta}). \end{matrix} \quad (19.21)$$

These formulas define the gauge transform

$$\varPhi_\beta \circ \varPhi_\alpha^{-1}\colon (t, x) \to (t, H_{\beta\alpha}(t)x), \quad (19.22)$$

with the same matrix as in (19.21). The collection of matrices $H = H_{\beta\alpha} = H_{\alpha\beta}^{-1} \in \mathrm{GL}(n, \mathcal{O}(U_{\alpha\beta})$ form a matrix cocycle defining the bundle τ_n.

Definition 19.22. The bundle $\tau_n\colon J^n(T) \to T$, defined by the trivializations (19.20) (or, equivalently, by the matrix cocycle (19.21)) is called the *n-jet bundle* over the base T.

Example 19.23. The line bundle defined by the cocycle $r_{\alpha\beta}$ is equivalent to the cotangent bundle $\mathbf{T}^*T$ over the base T. Indeed, consider an arbitrary meromorphic cochain $\{f_\alpha\}$ associated with a section of this bundle. This means that $f_\beta = r_{\alpha\beta} f_\beta$ on any intersection $U_{\alpha\beta}$. We claim that this cochain consistently defines a meromorphic 1-form ω by the rules

$$\omega_\alpha(D_\alpha) = f_\alpha, \qquad \omega_\alpha \in \varLambda^1(U_\alpha) \otimes \mathcal{M}(U_\alpha).$$

Indeed, on the overlapping $U_{\alpha\beta}$ the forms coincide, $\omega_\alpha = \omega_\beta$ (can be verified either on D_α or D_β), thus the cochain $\{\omega_\alpha\}$ defines a global meromorphic 1-form $\omega \in \Lambda^1(T) \otimes \mathbf{M}(T)$.

Example 19.24. On the Riemann sphere $T = \mathbb{P}$ the two fields $D_0 = \frac{\partial}{\partial t} \in \mathcal{D}(\mathbb{C})$ and $D_1 = t^2 \frac{\partial}{\partial t} \in \mathcal{D}(\mathbb{P} \smallsetminus \{0\})$ define the Birkhoff–Grothendieck cocycle corresponding to the bundle $\tau_n(\mathbb{P})\colon J^n(\mathbb{P}) \to \mathbb{P}$ with the corresponding function $r_{10}(t) = t^2$. The determinant bundle $\det \tau_n$ is associated with the cocycle $\det H_{10} = t^{n(n+1)}$. Thus the degree of the bundle is nonzero,

$$\deg \tau_n = -n(n+1) \neq 0 \qquad \text{for } n \geqslant 1, \tag{19.23}$$

and hence the jet bundle is nontrivial for all $n \geqslant 1$. For $n = 0$ the bundle τ_0 is obviously trivial, $J^0(T) = T \times \mathbb{C}^{n+1}$ for any base T. The 1-jet bundle is described in Problem 19.8.

Every meromorphic function $u \in \mathbf{M}(T)$ defines a meromorphic section of the jet bundle $t \mapsto j^n_u(t)$, called the *jet extension* of u, which is holomorphic outside the polar locus of u. However, not every section of τ_n is the collection of jets of some function: there are integrability conditions that are necessary.

Let ω_α be holomorphic 1-forms dual to the vector fields D_α, $\omega_\alpha(D_\alpha) \equiv 1$. This is a holomorphic cochain of 1-forms. On any trivializing chart $\tau_n^{-1}(U_\alpha) \cong U_\alpha \times \mathbb{C}^{n+1}$, using the scalar form ω_α, we can construct a 2-dimensional distribution as the common null space of $n-1$ Pfaffian forms

$$dx_0 - x_1\omega_\alpha = 0,\ dx_1 - x_2\omega_\alpha = 0,\ \dots,\ dx_{n-1} - x_n\omega_\alpha = 0. \tag{19.24}$$

One can instantly verify that two such distributions defined over two different trivializations, are related by the same gauge transforms (19.21) (note that the formulas (19.24) “naively mean” that $D_\alpha x_k = x_{k+1}$).

Definition 19.25. The 2-dimensional distribution defined on the n-jet bundle $J^n(T)$ by the formulas (19.24) in the trivializing charts, is called the *Cartan distribution*.

The Cartan distribution singles out sections of the jet bundle, which are jet extensions of meromorphic functions. Namely, if $\mathcal{C}_q$ is the 2-dimensional subspace of the Cartan distribution at a point $q \in J^n(T)$ and $u \in \mathcal{O}(T, p)$ is a holomorphic germ at the point $p = \tau_n(q) \in T$ such that $j^n_u(p) = q$, then the graph of the section $t \mapsto j^n_f(t)$ is a holomorphic curve tangent to the plane $\mathcal{C}_q$. Moreover, one can easily verify that $\mathcal{C}$ can be “axiomatically” (invariantly) defined as the only 2-dimensional distribution on $J^n(T)$ which is tangent to graphs of all meromorphic sections of the form $t \mapsto j^n_u(t)$.

Conversely, any meromorphic section $s \in \Gamma(\tau_n)$ whose graph is tangent to the Cartan distribution at all points, is the graph of a jet extension of a meromorphic function $u \in \mathcal{M}(T)$, $s = j^n_u$. In the future we will refer to sections tangent to the Cartan distribution as the *integrable sections*.

Finally we make the following obvious observation: the bundles $J^n(T)$ are naturally "nested", more precisely, there exist bundle maps (all fibered over the identity) making the following diagram commutative:

$$\begin{array}{ccccccccc} J^0(T) & \xleftarrow{\tau^1_0} & J^1(T) & \xleftarrow{\tau^2_1} & \cdots & \xleftarrow{\tau^n_{n-1}} & J^n(T) & \xleftarrow{\tau^{n+1}_n} & \cdots \\ \tau_0 \downarrow & & \tau_1 \downarrow & & & & \downarrow \tau_n & & \\ T & = & T & = & \cdots = & & T & = & \cdots \end{array} \tag{19.25}$$

The maps τ^k_{k-1} simply "forget" the last derivative. The kernel of each such map is one-dimensional. The corresponding one-dimensional subbundle $\mathcal{V}_k \subset J^k(T)$ will be referred to as *vertical* subbundle.

Now everything is ready to define in invariant terms linear ordinary differential equations.

Theorem 19.26. *For any holomorphic subbundle $\mathcal{L} \subset J^n(T)$ of codimension 1 in the n-jet bundle, transversal to the vertical subbundle $\mathcal{V} = \ker \tau^n_{n-1}$ almost everywhere, there exists a meromorphic connexion $\nabla = \nabla_{\mathcal{L}}$ on $J^n(T)$ with the following properties*:

1. *the subbundle $\mathcal{L}$ is invariant by ∇,*
2. *the singular locus of $\Sigma = \operatorname{Sing} \nabla$ consists of the points where $\mathcal{L}$ is nontransversal to the vertical bundle $\mathcal{V}$,*
3. *all ∇-horizontal sections of τ_n are integrable, i.e., are graphs of n-jet extensions of functions on T.*

The restriction of ∇ on $\mathcal{L}$ is uniquely defined.

Proof. The Cartan distribution restricted on the subbundle $\mathcal{L}$ (holomorphic submanifold of codimension 1) induces a 1-dimensional distribution (line field) on this bundle, eventually with singularities at the points of nontransversality between $\mathcal{L}$ and $\mathcal{C}$. The Cartan distribution always contains the vertical direction, hence transversality to $\mathcal{V}$ implies transversality to $\mathcal{C}$. Because of one-dimensionality, the constructed distribution is integrable. The integral curves (leaves of the integral foliation) by construction are tangent to the Cartan distribution $\mathcal{C}$. It remains to verify that the leaves of this foliation on $\mathcal{L}$ are horizontal sections for some meromorphic connexion ∇ on $J^n(T)$. We will explicitly construct the $(n+1) \times (n+1)$-matrix connexion 1-form Ω in any trivialization of $J^n(T)$, defined by a nonsingular vector field $D \in \mathcal{D}(U)$, $U \subseteq T$, or the dual form $\omega \in \Lambda^1(U) \otimes \mathcal{M}(U)$, as in (19.20).

The subbundle $\mathcal{L}$ in this trivializing chart is defined by a holomorphic equation $\sum_0^n a_i(t) x_{n-i} = 0$. Its differential (the tangent hyperplane to $\mathcal{L}$) modulo the Pfaffian equations (19.24) which define the Cartan distribution, is equal to

$$a_0\, dx_n + x_n(da_0 + a_1\omega) + x_{n-1}(da_1 + a_2\omega) + \cdots + x_1(da_{n-1} + a_n\omega) + x_0\, da_n.$$

If outside the singular locus $\{a_0 = 0\} \cap U$ this Pfaffian equation can be resolved with respect to dx_n. In conjunction with the Cartan equations this yields a meromorphic linear system over U, which by construction is tangent to the hypersurface $\{\sum_0^n a_i x_{n-i} = 0\}$ and the Cartan distribution. □

The connexion constructed in Theorem 19.26 is *not a companion connexion* on $J^n(T)$: its only advantage is the invariant construction. In practice the bundle $\mathfrak{L}$ satisfying the assumptions of the theorem, is projected along the vertical direction onto the $(n-1)$-jet bundle. The projection τ^n_{n-1} restricted on $\mathfrak{L}$, is a *meromorphic bundle map*, which carries the connexion $\nabla|_{\mathfrak{L}}$ to the meromorphic connexion defined by the Pfaffian companion system (19.17) with $b_i = -a_i/a_0$: the last equation is obtained by resolving the linear identity $\sum a_i x_{n-i} = 0$ with respect to x_n and substituting the result in the last Cartan equation $dx_{n-1} = \omega x_n$. Thus $\rho^n_{n-1}|_{\mathfrak{L}}$ carries $\nabla|_{\mathfrak{L}}$ into the companion connexion on $J^{n-1}(T)$.

For arbitrary (not regular) equations their interpretation as a connexion tangent to a subbundle $\mathfrak{L} \subset J^n(T)$ is as good (or as bad) as any other connexion meromorphically equivalent to it, in particular, as the companion connexion on the bundle $J^{n-1}(T)$ associated with an arbitrary meromorphic vector field $D \in \mathfrak{D}(T)$ or the corresponding dual form $\omega \in \Lambda^1(T) \otimes \mathbf{\mathfrak{M}}(T)$. The "naive approach" described in §19**B**, corresponds to the choice of $D = \frac{\partial}{\partial t} \in \mathfrak{D}(\mathbb{P})$ (note that the bundle $J^{n-1}(T)$ is also nontrivial, and this choice of D does not properly address the presence or absence of singularities at infinity).

However, if the connexion is *regular*, then it is natural to look for a bundle with *Fuchsian connexion* on it, *meromorphically* equivalent to the bundle $\mathfrak{L} \subset J^n(T)$ with the connexion $\nabla_{\mathfrak{L}}$.

Theorem 19.27. *If $L \in \mathfrak{LD}(\mathbb{P})$ is an arbitrary differential operator such that the linear equation $Lu = 0$ has $m \geqslant 0$ regular singularities, then the meromorphic connexion $\nabla|_{\mathfrak{L}}$ constructed in Theorem 19.26 is meromorphically gauge equivalent to a Fuchsian connexion on a holomorphic vector bundle π of rank n over $\mathbb{P}$. The degree of this bundle is equal to $(m-2)n(n-1)/2$.*

Proof. The existence of a Fuchsian connexion on an abstract bundle follows from the fact that any regular singularity at $t = t_j \in \mathbb{P}$ becomes Fuchsian after the local meromorphic gauge transform (re-expanding L in powers of $(t-t_j)\frac{\partial}{\partial t}$ rather than in powers of $\frac{\partial}{\partial t}$) by Theorem 19.20.

If $m = 2$, then there exists a holomorphic vector field D on $\mathbb{P}$ with exactly two simple (hyperbolic) singularities at two specified points. Expanding L in powers of D, we obtain expansion with holomorphic (hence constant) coefficients and nonvanishing leading term. Such an equation is necessarily an Euler equation (Problem 19.12) on the trivial bundle over $\mathbb{P}$.

If $m \neq 2$, such a vector field does not exist and the resulting bundle will be nontrivial. Assume that the point at infinity is nonsingular for the equation $Lu = 0$, and denote by $t_1, \dots, t_m \in \mathbb{C}$ distinct singular points of the equation, $\max_j |t_j| < R$. Consider two meromorphic vector fields on $\mathbb{P}$,

$$D_0 = \prod_{j=1}^{m} (t - t_j) \frac{\partial}{\partial t}, \qquad D_1 = t^{2-m} D_0.$$

They are holomorphic in the respective domains $U_0 = \mathbb{C}$, $U_1 = \mathbb{P} \smallsetminus \{|t| < R\}$ of the standard Birkhoff–Grothendieck covering and have singularities ("roots") only at the singular points of the equation.

By Theorem 19.20, after expansion in powers of D_0, D_1 and reduction to the corresponding companion form, we will obtain two meromorphic matrix functions Ω_0, Ω_1, with the following properties:

(1) Ω_0 has only Fuchsian singularities (simple poles) at the points $t_1, \dots, t_m$ and holomorphic at all other points of U_0,

(2) Ω_1 is holomorphic in U_1,

(3) in the annulus U_{01} the two forms are conjugated by the matrix function $H = H_{10}(t)$ as in (19.21) with the function $r = r_{10}(t) = t^{2-m}$.

The determinant $\det H_{10} = t^{(2-m)n(n-1)/2} = \det H_{01}^{-1}$ is the standard cocycle associated with the bundle ξ_d, $d = (m-2)n(n-1)/2$. Hence Ω_0, Ω_1 are two trivializations of a Fuchsian connexion on the holomorphic vector bundle associated with the cocycle $\{H_{01}, H_{10}\}$, which has degree d. □

From this result and Corollary 17.35 we immediately derive the assertion on the sum of all characteristic exponents.

Corollary 19.28. *The total of all characteristic exponents of a regular equation of order n with m singular points is equal to $(m-2)n(n-1)/2$.* □

19F. Riemann–Hilbert problem for higher order equations. The Riemann–Hilbert problem for scalar equations is to construct a Fuchsian equation of order n on $\mathbb{P}$ with the specified monodromy group. This problem is usually not solvable for one simple reason: the dimension of the variety of different monodromy data is larger than the dimension of the variety of Fuchsian equations.

Indeed, any equation with $m+2$ singular points $t_0 = 0, t_1, \dots, t_m \in \mathbb{C}$, $t_{m+1} = \infty \in \mathbb{P}$, is Fuchsian if and only if the corresponding linear operator can be written in the form

$$\begin{aligned} &L = D^n + a_1 D^{n-1} + \cdots + a_n, \qquad D = \tfrac{\partial}{\partial t}, \quad a_k = \frac{p_k(t)}{\Delta^k(t)}, \\ &p_k \in \mathbb{C}[t],\ \deg p_k \leqslant mk, \quad k = 1, \dots, n, \quad \Delta(t) = \prod_1^m (t - t_j) \end{aligned} \tag{19.26}$$

because of the restrictions on the order of the poles of coefficients at all singularities (note that D has a simple pole at both t_0 and t_{m+1}). The total number of parameters (assuming that the singular locus is fixed) is equal to

$$(m+1) + (2m+1) + \cdots + (nm+1) = \tfrac{1}{2}mn(n+1) + n.$$

The total number of entries in $m+1$ monodromy matrices is $(m+1)n^2$, (the last matrix is uniquely defined by the requirement that the product is equal to identity). In fact, one can assume that one of the matrices is reduced to the Jordan normal form which involves n diagonal terms (and the discrete choice 0 or 1 for the above-diagonal sequence). Thus the variety of all monodromy data has dimension equal to $mn^2 + n$.

The second number is almost always greater than the first, thus the Riemann–Hilbert problem is not solvable for most monodromy data. The exceptional combinations when the equality occurs, are $m = 0$ and $n = 1$. The first case corresponds to Euler equations (Problem 19.12), the second to the scalar equation. In the second case the monodromy is commutative and clearly any collection of m multiplicators can be realized by a scalar first order equation with preassigned poles.

For the Euler equation the monodromy group is determined by a single matrix M.

Proposition 19.29. *Any invertible matrix $M \in \mathrm{GL}(n, \mathbb{C})$ can be realized (modulo conjugacy) as the monodromy matrix of an Euler operator*

$$D'^n + a_1 D'^{n-1} + \cdots + a_{n-1} D' + a_n, \qquad D' = t\tfrac{\partial}{\partial t}, \quad a_1, \ldots, a_n \in \mathbb{C}. \tag{19.27}$$

Proof. We will show how a matrix in the Jordan normal form can be realized by the monodromy of an Euler equation.

One can immediately verify that the monodromy matrix of the operator D'^k, $k \geqslant 1$, is the (maximal) nilpotent Jordan $k \times k$-block in the basis $1, \ln t, \ldots, \ln^{k-1} t$. The "conjugated" operator $(D' - \lambda)^k$ has the maximal Jordan block with the eigenvalue $\mu = \exp 2\pi i \lambda$ in the basis $t^\lambda \ln^j t$, $j = 0, 1, \ldots, k-1$.

To build an arbitrary matrix with several Jordan blocks of various sizes, we use the composition of elementary factors of this form, which is again a monic Euler operator. Note that the Euler operators are always commuting between themselves, since their coefficients are constant.

If M consists of several Jordan blocks of sizes $\nu_1, \ldots, \nu_s$ with *the same eigenvalue* $\mu \neq 0$, then this monodromy matrix is realized by the composition of *commuting* operators $L = \prod_{j=1}^{s} (D' - \lambda - j)^{\nu_j}$ for any fixed choice of the logarithm $\lambda = \frac{1}{2\pi i} \ln \mu$.

Finally, if $M = \operatorname{diag}\{M_1, \dots, M_r\}$ with the spectra M_j being singletons μ_j, then each block can be realized by an Euler operator L_{μ_j}, and the entire matrix is realized by the "product" (composition) of commuting operators $L = L_{\mu_1} \cdots L_{\mu_r}$. □

One can attempt to relax the Riemann–Hilbert problem for Fuchsian equations and demand less. For instance, the natural question would be whether one can realize a given collection of characteristic exponents by a suitable Fuchsian equation.

The "variety of exponents" of a Fuchsian system with m singularities has dimension $mn - 1$. This dimension is by one less than the product mn because the exponents are constrained by the equality from Corollary 19.28. Compared to the dimension of the variety of Fuchsian equations of the given order with the specified number of singularities, it is almost always less than the latter, which means that in general the solution should be nonunique.

There is only one case where the two dimensions coincide: $m = 3$, $n = 2$, i.e., for equations of second order with three singularities. The total sum of characteristic exponents in this case is equal to 1 by Corollary 19.28.

Theorem 19.30. *Any 6 numbers whose sum is equal to* 1*, can be realized as characteristic exponents of a Fuchsian equation of second order with three singular points.*

Proof. First we note that the characteristic exponents at each point can be shifted by an arbitrary constant, provided that these three constants added together give zero (Problem 19.16). Thus it is sufficient to realize the collection of exponents of the form

$$(0, \alpha), \quad (0, \beta), \quad \bigl(\gamma, 1 - (\alpha + \beta + \gamma)\bigr) \tag{19.28}$$

One can always use the method of indeterminate coefficients (19.26), expressing explicitly the characteristic exponents of this equation and show that the corresponding interpolation problem for polynomial coefficients indeed has a unique solution.

The freedom to choose the derivation allows us to reduce these computations very substantially. Assume (as is always done) that the three singularities are at the points $0, 1$ and ∞. Consider the vector field $D = t(t-1)\frac{\partial}{\partial t}$ which has simple singularities at $t = 0, 1$ with eigenvalues -1 and 1 respectively, and nonsingular point at infinity.

The operator

$$L = D^2 + p_1(t)\, D + q_2(t), \qquad D = t(t-1)\tfrac{\partial}{\partial t}, \tag{19.29}$$

is Fuchsian if p_1, q_2 are holomorphic functions in the entire finite part $\mathbb{C}$ with poles of respective orders at most 1 and 2 at infinity (Proposition 19.19).

This means that p_1 and q_2 are polynomials in t of the degrees 1 and 2 respectively.

The corresponding characteristic exponents at the points $t_0 = 0$ and $t_1 = 1$ are roots of the polynomials $(-\lambda)^2+p(t_0)(-\lambda)+q(t_0)$ and $\lambda^2+p(t_1)\lambda+q(t_1)$ respectively (changing λ to $-\lambda$ happens since the eigenvalue of D at t_0 is -1); see Example 19.21. Thus p is a linear polynomial taking values $-\alpha$ and β at the points $t_0 = 0$ and $t_1 = 1$ respectively, and q vanishes at both these points, $q = ct(t-1)$. To express the characteristic exponents at infinity, we re-expand the operator (19.29) in powers of the Euler operator $D' = (t-1)^{-1}D$ which has eigenvalue -1 at infinity. After division by $(t-1)^2$ we obtain a monic differential polynomial with the free term $ct/(t-1) \underset{t\to\infty}{\longrightarrow} c$, whose value at $t = \infty$ is equal to the product $\gamma_1\gamma_2$ of the characteristic exponents at the point $t_2 = \infty$.

Thus letting $c = \gamma(1-(\alpha+\beta+\gamma))$ we obtain the *hypergeometric equation* which solves the “relaxed Riemann–Hilbert problem” in the specific case of second order and three singularities,

$$L = D^2+\bigl(-\alpha+t\beta)\bigr)\,D+\gamma\bigl(1-(\alpha+\beta+\gamma)\bigr)\,t(t-1), \quad D = t(t-1)\tfrac{\partial}{\partial t}. \tag{19.30}$$

This expansion can be more easily memorized than the standard expansion [**Inc44**] of the hypergeometric equation

$$L = t(1-t)D'^2 + \bigl(\gamma' - (\alpha'+\beta'+1)t\bigr)D' - \alpha'\beta', \qquad D' = \tfrac{\partial}{\partial t}, \tag{19.31}$$

which has characteristic exponents at the same three points $0, 1, \infty$ equal to

$$(0, 1-\gamma'), \quad (0, \gamma'-\alpha'-\beta'), \quad (\alpha', \beta').$$

The old-fashioned name for a general solution of this equation is the *Riemann P-function.* □

Remark 19.31. The term “hypergeometric *system*” is reserved for linear systems on $\mathbb{P}$ of a special form. Let $S \in \operatorname{Mat}(n, \mathbb{C})$ be a diagonalizable matrix with simple spectrum $\{s_, \dots, s_m\}$, and $A \in \operatorname{Mat}(n, \mathbb{C})$ an arbitrary matrix. Consider the linear system associated with the ordinary differential equation

$$(tE - S)\dot{x} = Bx, \qquad x \in \mathbb{C}^n,\ t \in \mathbb{C} \subset \mathbb{P}, \tag{19.32}$$

where E stands for the identical matrix. By a linear change of variables the matrix S can always be diagonalized. After inversion we have the meromorphic system

$$\dot{x} = \begin{pmatrix} (t-s_1)^{-1} & & \\ & \ddots & \\ & & (t-s_n)^{-1} \end{pmatrix} Bx. \tag{19.33}$$

This system has simple poles at the points $s_1, \dots, s_n$ and at the point $t = \infty$. The residue matrix A_j at each point has rank 1: the only nonzero row of this matrix is the jth row of the matrix B. Therefore the characteristic exponents at this point are all zeros, eventually except for the value $b_{jj} \in \mathbb{C}$.

The bridge between two notions, the hypergeometric systems and hypergeometric equations, is obvious. Each component of the hypergeometric 2×2-system

$$\begin{pmatrix} t & \\ & t-1 \end{pmatrix} \dot{x} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} x. \tag{19.34}$$

satisfies a hypergeometric equation (19.30) (Problem 19.17).

Exercises and Problems for §19.

Problem 19.1 ([**VK75**], [**Kra97**]). Prove that a $\mathbb{C}$-linear self-map $L\colon \mathcal{M} \to \mathcal{M}$ is a linear differential operator of order $\leqslant n$, if and only if the iterated commutator $[g_0, [g_1, [\cdots, [g_n, L] \cdots]]]$ vanishes identically as a self-map of $\mathcal{M}$ for any $n+1$ multiplications $g_i\colon \mathcal{M} \to \mathcal{M}$, $f \mapsto g_i f$.

Exercise 19.2. Prove that the monodromy of a linear equation $Lf = 0$, $L \in \mathfrak{L}\mathcal{O}(T)$, is reducible if and only if the holonomy of the respective companion system is reducible.

Problem 19.3. Let $f_1, \dots, f_n$ be functions holomorphic in a domain $U \subset T$. Prove that if $W(f_1, \dots, f_n) \equiv 0$, then these functions are linearly dependent over $\mathbb{C}$. Is this true for C^∞-smooth functions?

Exercise 19.4. Prove in detail Proposition 19.19.

Problem 19.5. Find characteristic exponents at the origin for a Fuchsian operator $L = D^n + a_1 D^{n-1} + \cdots + a_n$ with holomorphic coefficients $a_k \in \mathcal{O}(\mathbb{C}, 0)$ and a holomorphic vector field $D = (ct + \cdots)\frac{d}{dt}$ with $c \neq 0$.

Problem 19.6. Prove that the point $t_0 = \infty$ is Fuchsian for the monic linear operator (19.2) expanded in the powers of $D = \frac{\partial}{\partial t}$ with $a_0 \equiv 1$, if and only if $\operatorname{ord}_\infty a_k \geqslant k + 2 - n$.

Exercise 19.7. Let $s = \lambda_1 + \cdots + \lambda_n$ be the sum of characteristic exponents of a regular singularity of a linear equation $Lf = 0$. Prove that the Wronskian of a fundamental system of solutions $w(t) = W(f_1, \dots, f_n)$ can be represented as $w(t) = t^{s+n(n-1)/2} h(t)$, $h \in \mathcal{O}(\mathbb{C}, 0)$, $h(0) \neq 0$.

Problem 19.8. Prove that the 1-jet bundle $J^1(T)$ is equivalent to the direct sum of the trivial bundle of rank 1 and the cotangent bundle $\mathbf{T}^*T$ for any base T.

Problem 19.9. Let $\mathcal{C}'$ be a holomorphic 2-distribution on the jet bundle, which is tangent to graphs of all sections of the form $t \mapsto j_u^n(t)$ for all holomorphic germs $u \in \mathcal{O}(T, p)$, $p \in T$.

Prove that $\mathcal{C}'$ coincides with the Cartan distribution.

Problem 19.10. Prove that any integrable section $s \in \Gamma(\tau_n)$ of the jet bundle τ_n, is the jet extension of a meromorphic function $u \in \mathcal{M}(T)$, i.e., $s = j_u^n$.

Problem 19.11. Prove that the Cartan distribution itself is nonintegrable in the sense of Theorem 2.9.

Problem 19.12. Prove that a linear equation of order n with two regular singularities at $t = 0$ and $t = \infty$ is an Euler equation, i.e., it has the form

$$Lu = 0, \quad L = D^n + a_1 D^{n-1} + \cdots + a_{n-1} D + a_n, \quad a_j \in \mathbb{C},\ D = t\tfrac{\partial}{\partial t}.$$

Find the complete factorization of the Euler equation into composition of first order Fuchsian operators.

Problem 19.13. Prove that for a regular linear equation with m singular points on a compact Riemann surface T, the sum of all characteristic exponents is equal to $(m-\chi)n(n-1)/2$, where $\chi = \deg \mathbf{T}^*T$ is the Euler characteristic (the degree of the cotangent bundle).

Exercise 19.14. Let $D = t\frac{\partial}{\partial t}$ be the Euler operator and u is the "operator of multiplication by t^λ", $\lambda \in \mathbb{C}$. Prove that the conjugated operator $u^{-1}Du$ is again a first order with meromorphic coefficients. Compute it.

Exercise 19.15. Let u be the operator of multiplication by a germ $c(t-t_0)^\lambda h(t)$, $h \in \mathcal{O}(\mathbb{C}, t_0)$, $h(t_0) \neq 0$. Prove that there exists a holomorphic vector field $D \in \mathcal{D}(\mathbb{C}, t_0)$ with a simple (hyperbolic) singular point, such that $u^{-1}Du = D + \lambda$ (cf. with the previous exercise).

Problem 19.16. Show that for an arbitrary Fuchsian operator L of order n with singularities at the points $t_1, \dots, t_m \in \mathbb{P}$ and arbitrary collection of the complex numbers $\lambda_1, \dots, \lambda_m$ such that $\sum \lambda_j = 0$, one can find another Fuchsian operator L' with the same singular points, such that the characteristic exponents $\alpha_{1,j}, \dots, \alpha_{n,j}$ at each singular point t_j are shifted by λ_j: $\alpha'_{i,j} = \alpha_{i,j} + \lambda_j$ for all i, j.

Problem 19.17. Find explicitly the hypergeometric equation (19.30) and the corresponding characteristic exponents for each component of the system (19.34).

20. Irregular singularities and the Stokes phenomenon

Unlike the Fuchsian singularities which can always be reduced to a simple formal normal form by means of a convergent gauge transform, irregular singularities have the formal classification considerably more involved and the normalizing transformations as a rule diverge.

20A. One-dimensional irregular singular points. Irregular singularities of scalar (one-dimensional) linear equations admit complete investigation. Consider the equation

$$t^m \dot{x} = a(t)x, \qquad m \geqslant 2, \quad a(t) = \lambda + a_1 t + a_2 t^2 + \cdots \in \mathcal{O}(\mathbb{C}, 0). \tag{20.1}$$

Its nontrivial solution is given by the explicit formula

$$x(t) = \exp \int \frac{a(t)}{t^m}\, dt = \exp[-t^{1-m}\lambda(1 + o(1))]. \tag{20.2}$$

The origin is an essential singularity of the function $x(t)$ holomorphic in the punctured neighborhood $(\mathbb{C}, 0) \smallsetminus \{0\}$.