

Exercise 17.17. Prove that the degree of the bundle ξ_D is equal to $|D| = d_1 + \dots + d_n$.

Problem 17.18. Prove that any holomorphically invertible matrix function $F(t)$ in the annulus $A = U_0 \cap U_1$ can be factored out as $F(t) = H_0(t)H_1(t)t^D$ with the terms $H_i(t)$ holomorphically invertible in U_i , $i = 0, 1$, and an integer diagonal matrix D . It is this form that is sometimes called the *Birkhoff factorization*.

In particular, any nonzero meromorphic germ of a matrix function $F(t)$ at the infinity admits factorization $F(t) = \Pi(t)H(t)t^D$ with a monopole $\Pi(t)$ and a holomorphically invertible germ $H(t)$ at infinity.

Problem 17.19 ([Bol00]). Prove that a holomorphic vector bundle $\pi: S \rightarrow T$ is *topologically* trivial if and only if its degree is equal to zero. The topological triviality means that there exists a *homeomorphism* $F: S \rightarrow T \times \mathbb{C}^n$ fibered over the identity and linear on each fiber.

18. Riemann–Hilbert problem

The problem is as follows: *To show that there always exists a linear differential equation of the Fuchsian class, with given singular points and monodromy group.* The problem requires the production of n functions of the variable z , regular throughout the complex z -plane except at the given singular points; at these points the functions may become infinite of only finite order, and when z describes circuits about these points the functions shall undergo the prescribed linear substitutions.

D. Hilbert, 1901, reprinted from [Hil00]

The *Riemann–Hilbert problem*, also known as Hilbert’s twenty-first problem, requires constructing a linear system with the prescribed monodromy group and positions of all singularities. The original formulation by Hilbert is somewhat confusing, since the clarification given in the text after it, describes only the regularity condition, while the main formulation explicitly mentions Fuchsian systems.

One can think of not one but rather *three* different accurate formulations, when a given monodromy group is required to be realized by:

- (i) a Fuchsian linear n th order differential equation,
- (ii) a linear system having only regular singularities, or
- (iii) a Fuchsian system on the whole Riemann sphere \mathbb{P} .

In each case it is required that the equation (resp., the system) be nonsingular outside the preassigned points.

The negative answer in the first problem was known already by A. Poincaré: the reason is that the dimension of the space of all Fuchsian equations having m prescribed singular points on \mathbb{P} , is strictly smaller than the dimension of all admissible monodromy data, except for the case of second order

equations with three singular points studied by Riemann. The corresponding problem is discussed in §19F.

J. Plemelj [Ple64] gave a solution of problem (ii) while claiming solution of the strongest problem (iii). The gap was discovered by Yu. Ilyashenko [AI85] and A. Treibich [Tre83] in the earlier eighties. The positive part of the Plemelj theorem is described in §18B.

Yet only recently it became clear that there is a substantial difference between the formulations (ii) and (iii). It was proved independently by A. Bolibruch [Bol92] and V. Kostov [Kos92] that an irreducible monodromy group can always be realized by a Fuchsian system. In this section we explain a remarkably simple proof of the Bolibruch–Kostov theorem which was communicated to us by the late A. Bolibruch.

However, for a reducible monodromy group the answer to problem (iii) may be negative. The counterexample, also due to Bolibruch, is described in §18E.

The way to understand reasons and obstructions for solvability of the Riemann–Hilbert problem passes through its generalization, the Riemann–Hilbert problem for meromorphic connexions on holomorphic vector bundles. The “elementary” (analytic) demonstration of these results, is given in 16.

18A. Riemann–Hilbert problem on abstract bundles. In invariant terms the Riemann–Hilbert problem can be formulated as follows: construct a meromorphic connexion on the *trivial* bundle over the Riemann sphere, having preassigned Fuchsian singular points at the specified points and the preassigned holonomy group.

In the category of abstract vector bundles the Riemann–Hilbert problem becomes in a sense trivial: *any collection of matrix connexion forms* can be realized by a meromorphic connexion on a suitable holomorphic vector bundle.

We start by choosing a special system of generators for the monodromy group. Consider m distinct points a_1, \dots, a_m on the affine plane $\mathbb{C} \subset \mathbb{P}$. By choosing an appropriate affine chart one can always guarantee that $a_i/a_j \notin \mathbb{R}_+$, i.e., that the line segments $[0, a_j]$ connecting the origin with these points are pairwise disjoint except for the common origin.

Definition 18.1. The *canonical loops* generating the fundamental group of the Riemann sphere with finitely many deleted points $\mathbb{P} \setminus \Sigma$, $\Sigma = \{a_1, \dots, a_m\}$ are the loops which consist of line segments connecting the origin with each singular point $a_i \in \mathbb{C}$, encircling the latter along a small counterclockwise circular arc and then returning along the same segment in the opposite direction; see Fig. III.2. All circular arcs are pairwise disjoint.

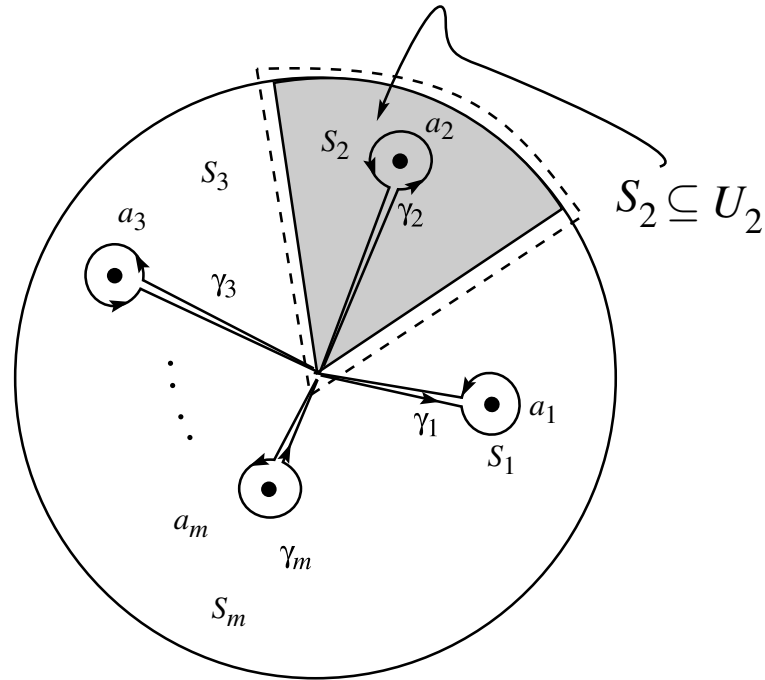


Figure III.2. Canonical loops and specification of the monodromy data

The fundamental group $\pi_1(\mathbb{P} \setminus \Sigma, 0)$ is generated by the canonical loops γ_i , $i = 1, \dots, m$, related by a single identity $\gamma_1 \circ \dots \circ \gamma_m = \text{id}$. We will always assume that the points are numbered counterclockwise (see Fig. III.2) and cyclically, i.e., the point a_m follows after a_{m-1} and is in turn followed by a_1 . Denote by $U_0 \subseteq \mathbb{C}$ the disk $\{|t| < R\}$ containing all points a_j .

Recall that the monodromy data is a collection of m points a_1, \dots, a_m as above and invertible linear operators $M_1, \dots, M_m \in \text{GL}(n, \mathbb{C})$ such that their product in the specified order is the identity; see (16.16).

Definition 18.2. The monodromy data is *realized* by a meromorphic connection ∇ on a holomorphic vector bundle of rank n over \mathbb{P} , if the singular points of the connection are a_j , $j = 1, 2, \dots, m$, and the holonomy Δ_j (the linear self-map of the fiber $\tau_0 = \pi^{-1}(0) \cong \mathbb{C}^n$ associated with each canonical loop γ_j) coincides with M_j for all $j = 1, \dots, m$.

Example 18.3 (Realization of a single operator). Every single holonomy operator can be immediately realized by the holonomy of a Fuchsian system. Indeed, let $U = U_j \subset \mathbb{C}$ be a simply connected domain containing both the origin and the point a_j . Then the holonomy operator for the Fuchsian matrix

1-form

$$\Omega_j = \frac{A_j dt}{t - a_j} \in \Lambda^1(U_j) \otimes \mathcal{M}(U_j), \quad \exp 2\pi i A_j = M_j, \quad (18.1)$$

associated with the fiber $\{0\} \times \mathbb{C}^n$, coincides with M_j . Recall that the equation $\exp 2\pi i A_j = M_j$ is solvable for any nondegenerate matrix M_j by Lemma 3.11.

Note that the realization is by no means unique: besides the freedom of choice for the matrix logarithm discussed in §3D, one can also construct (in the resonant case) a non-Euler system.

We show now how an arbitrary monodromy data for several singularities can be realized as a holonomy of a Fuchsian connexion on an abstract bundle. Consider a collection of meromorphic matrix 1-forms Ω_j , $j = 1, \dots, m$, such that each form is meromorphic in U_0 and has a unique pole at a_j . The collection $\{\Omega_j\}_1^m$ of such meromorphic matrix 1-forms will be called *admissible*, if $\Delta_m \circ \dots \circ \Delta_1 = \text{id}$. This happens automatically if each Ω_j realizes the holonomy operator M_j from the monodromy data $\{M_1, \dots, M_m\}$.

Theorem 18.4. *For any admissible collection of meromorphic 1-forms*

$$\begin{aligned} \Omega_j \in \Lambda^1(U_0) \otimes \mathcal{M}(U_0), \quad \text{Sing } \Omega_j = \{a_j\}, \quad j = 1, \dots, m, \\ \Delta_j = \Delta_{\gamma_j} \in \text{GL}(\tau_0), \quad \Delta_m \circ \dots \circ \Delta_1 = \text{id}, \end{aligned} \quad (18.2)$$

there exists a holomorphic vector bundle $\pi: S \rightarrow \mathbb{P}$ of rank n over the Riemann sphere and a meromorphic connexion ∇ on this bundle such that the singular locus of ∇ coincides with $\Sigma = \{a_1, \dots, a_m\}$ and at each singular point a_k the connexion is locally biholomorphically equivalent to the connexion $d - \Omega_k$.

In other words, one can construct holomorphic bundles over \mathbb{P} with any preassigned holonomy group, specifying in addition the types of singularities (regular, Fuchsian or even arbitrary irregular) as well as their position. Of course, there is no guarantee that the bundle obtained this way, will be trivial.

Proof. The assertion of the theorem is largely a tautology very similar to that asserted in Theorem 17.6. The accurate proof consists of two steps.

On the first step we construct a holomorphic bundle $\pi: S \rightarrow U_0$ over the large disk U_0 and a meromorphic connexion on it with the specified holonomy operators. Because of the admissibility, the holonomy associated with the boundary of the disk is identical. On the second step we “seal” the hole at infinity, constructing a holomorphic vector bundle over \mathbb{P} .

We construct explicitly the cocycle which defines the bundle π over the disk U_0 as follows. To define the covering, we slice the disk into sectors

$S_j = \{\alpha_j \leq \text{Arg } t \leq \alpha_{j+1}, |t| < R\}$ in such a way that each sector S_j contains only one singular point, and consider the covering of U_0 by the open domains $U_j, j = 1, \dots, m$, which are small ε -neighborhoods of these sectors. The number ε is chosen so small that the intersections $U_{j,j+1} = U_j \cap U_{j+1}$, the ε -neighborhoods of the rays $\text{Arg } t = \alpha_j$, are all disjoint from the singular locus Σ . Note that the origin $t = 0$ belongs to all domains U_j .

If we slit each domain U_j along the radius connecting the corresponding point a_j with the boundary of the disk U_0 , then none of these slits intersect the pairwise intersections U_{ij} . On the other hand, in the slit domains we may define holomorphic invertible matrix solutions $X_j(t)$ of the matrix differential equations $dX_j = \Omega_j X_j$ with the initial condition $X_j(0) = E$.

Define the holomorphic matrix cocycle

$$H_{ij} = X_i \cdot X_j^{-1} \quad \text{on } U_{ij}. \quad (18.3)$$

The cocycle identities are obviously satisfied, and differentiating (18.3), we conclude that

$$dH_{ij} = dX_i \cdot X_j^{-1} + X_i(-X_j^{-1}dX_j \cdot X_j^{-1}) = \Omega_i H_{ij} - H_{ij} \Omega_j.$$

Let $\pi: S \rightarrow U_0$ be the holomorphic vector bundle $\pi: S \rightarrow U_0$ over the disk U_0 , described in Theorem 17.6, for which the cocycle $\mathcal{H} = \{H_{ij}\}$ is the collection of transition maps. Then the collection of the matrix forms Ω_i defines a meromorphic connexion ∇ on S with the polar locus Σ . Since $H_{ij}(0) = E$, the holonomy maps of this connexion, associated with the section $\pi^{-1}(0)$ and the loops γ_j , coincide with the prescribed linear operators Δ_j . In particular, the holonomy of the boundary circumference of the disk U_0 is trivial by (18.2).

To “seal the gap” and extend the bundle π just constructed over U_0 on the disk $\mathbb{P} \setminus U_0$, we consider the trivial bundle of the same rank n over the disk $U_1 = \{|t| > R - \varepsilon\} \subset \mathbb{P}$ on the Riemann sphere, equipped with the trivial connexion $\nabla = d$.

Any linear invertible map of a fiber $\pi^{-1}(a) \rightarrow \mathbb{C}^n, a \in U_{01} = U_0 \cap U_1$, can be extended uniquely as a holomorphic gauge map $H_{01}: \pi^{-1}(U_{01}) \rightarrow U_{01} \times \mathbb{C}^n$ fibered over the identity map of the annulus U_{01} , which sends horizontal sections of ∇ to the horizontal (constant) sections of the trivial bundle. In a standard way we can now construct the holomorphic bundle over the union $U_0 \cup U_1 = \mathbb{P}$ with a holomorphic connexion on it, without singularities outside U_0 and the prescribed holonomy group in U_0 . \square

18B. Connexions on the trivial bundle. If the abstract bundle $\pi: S \rightarrow \mathbb{P}$ constructed in Theorem 18.4 is *holomorphically* equivalent to the trivial bundle $\pi_0: \mathbb{P} \times \mathbb{C}^n \rightarrow \mathbb{P}$, the globally defined connexion matrix would solve

the Riemann–Hilbert problem in the classical sense. However, this holomorphic triviality may be only accidental, and in general the bundle will be nontrivial.

Nevertheless we can assume that the bundle is already in the standard Birkhoff–Grothendieck normal form, i.e., a pair of trivializations is chosen so that the transition cocycle between them is the standard matrix t^D .

For such a standard bundle we will construct an explicit *meromorphic trivialization*, a bundle map $F: S \rightarrow \mathbb{P} \times \mathbb{C}^n$ with a single pole at infinity. This bundle map is given by the cochain $\mathcal{F} = \{\text{id}, t^{-D}\}$, where D is the splitting type.

The trivializing map F carries the connexion ∇ on S to a meromorphic connexion on the trivial bundle over \mathbb{P} . The resulting connexion has the same holonomy group, yet its singularity at infinity will in general only be regular non-Fuchsian.

The bundle π constructed in Theorem 18.4 is holomorphically equivalent to the standard Birkhoff–Grothendieck bundle, the equivalence being defined by a holomorphic matrix cochain $\mathcal{G} = (G_0, G_1)$. The meromorphic bundle map F which trivializes π , is the composition $\{F_0 = G_0, F_1 = t^{-D}G_1\}$, whose components are columns of the commutative diagram

$$\begin{array}{ccc}
 U_0 \supset A & \xrightarrow{H} & A \subset U_1 \\
 G_0 \downarrow & & \downarrow G_1 \\
 U_0 \supset A & \xrightarrow{t^D} & A \subset U_1 \\
 \text{id} \downarrow & & \downarrow t^{-D} \\
 U_0 \supset A & \xrightarrow{\text{id}} & A \subset U_1
 \end{array}
 \quad
 \begin{array}{l}
 A = U_0 \cap U_1, \\
 \mathbb{P} = U_0 \cup U_1, \\
 G_0 \in \text{GL}(n, \mathcal{O}(U_0)), \\
 G_1 \in \text{GL}(n, \mathcal{O}(U_1)).
 \end{array}
 \tag{18.4}$$

The upper square of this diagram is the holomorphic equivalence of the bundle π and the standard bundle ξ_D , the lower square is the meromorphic trivialization.

The Fuchsian connexion ∇ on the bundle π constructed in Theorem 18.4, is F -related with a connexion ∇_0 on the trivial bundle π_0 . Yet this connexion is obviously regular, even if the point $t = \infty$ was singular (Fuchsian) for ∇ . This immediately implies the affirmative solution of Problem (ii) (p. 312).

Theorem 18.5. *Any monodromy group can be realized by a regular linear system on the Riemann sphere.*

Moreover, the regular system can always be constructed with all singularities Fuchsian, except for at most one. □

Somewhat embarrassingly, the singularity of ∇_0 created at the point $t = \infty$, is non-Fuchsian even if this point was nonsingular for ∇ . Yet the regular

singular point at infinity for the connexion ∇_0 sometimes can be further simplified using monopole gauge transforms. Recall that the *monopole gauge transforms* are meromorphic gauge self-maps of the trivial bundle, which are nonsingular at all points of \mathbb{P} except the point $t = \infty$ (cf. Definition 16.34).

The following result was first proved⁴ by J. Plemelj in [Ple64].

Theorem 18.6. *If at least one monodromy operator M_j is diagonalizable, then the corresponding monodromy data can be realized by the holonomy of a Fuchsian system on \mathbb{P} .*

Proof. Consider the abstract bundle π realizing the specified holonomy group as in Theorem 18.4. Without loss of generality we may assume that the bundle is trivialized over two charts U_0, U_1 by a Birkhoff–Grothendieck cocycle H and the connexion ∇ is represented by two meromorphic matrix 1-forms Ω_0, Ω_1 . Again without loss of generality we may assume that the diagonalizable monodromy operator corresponds to the singular point $t = \infty$ and the corresponding Fuchsian connexion form over U_1 is already diagonal and is an Euler system in the standard chart,

$$dX = \Omega_1 X, \quad \Omega_1 = A \frac{dt}{t}, \quad A = \text{diag}\{\lambda_1, \dots, \lambda_n\}. \quad (18.5)$$

Consider the meromorphic gauge transform (18.4) which trivializes the bundle π . This transform carries ∇ to the meromorphic connexion ∇_0 on the trivial bundle, defined by the single meromorphic matrix 1-form $\Omega \in \text{Mat}(n, \mathcal{A}^1(\mathbb{P}) \otimes \mathcal{M}(\mathbb{P}))$ with all singularities in the finite plane already Fuchsian.

The singularity at infinity is regular non-Fuchsian and has a fundamental (multivalued) matrix solution of the form $X(t) = t^{-D} G_1 t^A$, as follows from the explicit form (18.5) and the diagram (18.4), where G_1 is a holomorphic invertible matrix function near $t = \infty$.

Inverting the order of matrix terms by Lemma 16.36, we may rewrite the solution $X(t)$ under the form

$$X(t) = \Pi^{-1}(t) G'(t) t^{-D'} t^A, \quad G' \in \text{GL}(n, \mathcal{O}(\mathbb{P}, \infty)), \quad \Pi \in \text{GL}(n, \mathbb{C}[t]).$$

After application of the monopole gauge transform Π we obtain a matrix form $\Omega' = d\Pi \cdot \Pi^{-1} + \Pi \Omega \Pi^{-1}$ with all finite singularities still Fuchsian (as the order of pole cannot be changed by a holomorphic local gauge equivalence) and the regular singular point at infinity, having a fundamental matrix solution $X'(t) = G'(t) t^{A-D'}$, since diagonal matrices commute and $t^{-D'} t^A = t^{A-D'}$.

⁴The assumption on diagonalizability was missing in [Ple64], as was noted by Ilyashenko and Treibich.

From this representation it follows immediately that Ω' has a first order pole at infinity with the principal term conjugate to $(A - D')\frac{dt}{t}$, that is, Ω' is Fuchsian also at infinity. \square

18C. Invariant subbundles and irreducibility. Solvability of the Riemann–Hilbert problem for an arbitrary monodromy data is determined to a very large extent by existence and structure of *invariant subspaces* of the holonomy.

Let $\pi: S \rightarrow T$ be an arbitrary holomorphic vector bundle with a meromorphic connexion ∇ on it.

Definition 18.7. A subbundle $L \subset S$ is called ∇ -invariant, if fibers of this subbundle are mapped into each other by all the horizontal transport operators.

In other words, L is invariant, if any parallel transport operator Δ_γ between two fibers τ_a, τ_b along any path γ connecting these points in $T \setminus \Sigma$, maps $L_a = L \cap \tau_a$ into $L_b = L \cap \tau_b$.

A subspace $\ell \subset \mathbb{C}^n$ is invariant by a linear group $\mathfrak{G} \subset \text{GL}(n, \mathbb{C})$, if it is invariant by all operators from the group. Obviously, if for a finitely generated group $\mathfrak{G} = \langle M_1, \dots, M_m \rangle$ it is sufficient to verify invariance only by the generators.

Since monodromy operators Δ_γ for all loops $\gamma \in \pi_1(T \setminus \Sigma, a)$ are a special class of parallel transport maps, any subbundle $L \subset S$ invariant for a meromorphic connexion generates the invariant subspace $\ell_a = L \cap \tau_a$ for the monodromy group, regardless of the type of singular points. The inverse statement is true only for regular connexions (cf. with Problem 18.4).

Proposition 18.8. *Let ∇ be a regular meromorphic connexion on a holomorphic bundle $\pi: S \rightarrow T$.*

If $\ell \subset \tau_a = \pi^{-1}(a)$ is a linear subspace (sub-fiber) invariant by all holonomy operators Δ_γ , $\gamma \in \pi_1(T \setminus \Sigma, a)$, then there exists a holomorphic subbundle $L \subset S$ invariant by ∇ and extending ℓ , so that $L \supset \ell$.

Proof. The only candidate for such a subbundle is the saturation of ℓ by horizontal sections. We show that this saturation is indeed a holomorphic subbundle of S , namely, it extends holomorphically at all regular singularities.

1. By parallel transport along a path connecting the base point a with any nonsingular point $t \notin \Sigma$, we can carry the subspace ℓ to a subspace $L(t)$ in the fiber $\pi^{-1}(t)$. The result of this transport does not depend on the choice of the path, since ℓ is invariant by all holonomy operators.

The subspaces $L(t)$, $t \notin \Sigma$, holomorphically depend on the base point: to see this locally near any point $b \in T \setminus \Sigma$, it is sufficient to choose a trivialization in which the connexion form is identically zero. In this trivialization $L(t)$ is independent of t .

2. It remains to prove that the subbundle L over $T \setminus \Sigma$, analytically extends to any singular point. This is a purely local problem that can be solved in a fixed trivialization $(\mathbb{C}, 0) \times \mathbb{C}^n$. Let $X(t)$ be a fundamental matrix solution of the corresponding linear system $dX = \Omega X$.

3. Consider first the case where the monodromy is trivial, i.e., $X(t)$ is a meromorphic matrix function. Without loss of generality we may assume that the subspace $L(t)$ is spanned by the first k columns (vector functions) of X . Our goal is to show that one can find some other k holomorphic vector functions, linearly independent for all $t \in (\mathbb{C}, 0)$.

If $k = 1$, then any meromorphic vector function $x_1(t)$ can obviously be uniquely represented as $x_1(t) = t^{\nu_1} y_1(t)$ with $y_1(\cdot)$ holomorphic and $y_1(0) \neq 0$. The function $y_1(t)$ spans the same subspace (line) and is holomorphic.

Assume that any k -dimensional meromorphic family of subspaces can be spanned by k holomorphic linearly independent vector functions. Making an additional holomorphic gauge transform, we may assume without loss of generality that these vector functions coincide with the first coordinate vector functions $y_1(t) = (1, 0, \dots, 0)^\top$, $y_2(t) = (0, 1, 0, \dots)^\top$, etc. Consider the meromorphic vector function $x_{k+1}(t)$. Without changing the subspace $L(t)$, we can replace it by another vector function $x'_k(t)$ whose first k coordinates are identically zero (subtracting a suitable linear combination of $x_1(t), \dots, x_k(t)$ with meromorphic coefficients). The vector function $x'_{k+1}(t)$ can again be uniquely represented as $x'_{k+1}(t) = t^{\nu_{k+1}} y_{k+1}(t)$ with $y_{k+1}(t)$ holomorphic and $y_{k+1}(0) \neq 0$. Since the first components of y_{k+1} are identically zeros, the vector functions y_1, \dots, y_{k+1} are linearly independent.

4. Assume now that the monodromy of the singular point is nontrivial and the linear space generated by the first $k < n$ columns of the fundamental matrix solution $X(t)$ is invariant. If these columns are arranged in the form of a rectangular $n \times k$ -matrix $Y(t)$, then the invariance means that for some constant $k \times k$ -invertible matrix M the result of analytic continuation of Y around the origin is $Y(t)M$. Choosing any matrix logarithm $A \in \text{Mat}(k, \mathbb{C})$ such that $\exp 2\pi i A = M$, we conclude that the matrix function $Z(t) = Y(t)t^{-A}$ is single-valued hence meromorphic at the origin. The columns of Z generate the same subspace as the columns of Y , thus by the previous arguments this subspace holomorphically depends on t at any regular singular point. \square

Definition 18.9. A meromorphic connexion on a holomorphic vector bundle is called *reducible*, if it admits a nontrivial invariant holomorphic subbundle. Otherwise the connexion is called *irreducible*.

From Proposition 18.8 it follows that a regular connexion is irreducible if and only if its holonomy group is irreducible as a linear representation of the fundamental group $\pi_1(T \setminus \Sigma, a)$. In other words, (ir)reducibility is the property of the holonomy rather than of the connexion itself.

Example 18.10. Let Ω be a rational matrix 1-form on \mathbb{P} defining a connexion on the trivial bundle over the Riemann sphere. If Ω has a block upper-triangular form, then the connexion $\nabla = d - \Omega$ is reducible. The corresponding invariant subbundle is the “constant” coordinate subbundle spanned by the first coordinate vectors.

Lemma 18.11. *Suppose that a rational $n \times n$ -matrix 1-form Ω on the Riemann sphere \mathbb{P} has $m \geq 1$ Fuchsian points and a regular non-Fuchsian point at the origin. Assume that locally near the origin the fundamental solution of the system admits representation*

$$X(t) = t^N Y(t), \quad N = \text{diag}\{\nu_1, \dots, \nu_n\}, \quad \nu_i \in \mathbb{Z},$$

where the multivalued matrix function $Y(t)$ is a fundamental solution for a Fuchsian singularity (so that $dY \cdot Y^{-1}$ has a first order pole at the origin) and ν_i are some integer numbers.

If the global monodromy group of the system is irreducible, then the difference between the numbers ν_i is explicitly bounded,

$$|\nu_i - \nu_j| \leq (m - 2)(n - 1), \quad \forall i, j = 1, \dots, n. \quad (18.6)$$

Proof. The Pfaffian matrix of the system locally near the origin has the form

$$\Omega = N t^{-1} dt + t^N \Omega' t^{-N},$$

where $\Omega' = dY \cdot Y^{-1}$ has a first order pole at the origin. Without loss of generality, we may assume that the entries of the integer diagonal matrix N are arranged in the nonincreasing order,

$$\nu_1 \geq \dots \geq \nu_n$$

(one can always permute the rows by a global constant gauge transformation that preserves the irreducibility).

The idea of the proof is rather transparent: if two consecutive numbers ν_k, ν_{k+1} differ too much, then the matrix 1-form Ω will have a corner filled by rational forms of bounded degrees which are too flat to be nonzero. On the other hand, a zero corner implies reducibility which is forbidden by the assumptions of the lemma.

More accurate reasoning is as follows. If $\nu_k - \nu_{k+1} > m - 1$ for some k between 1 and $n - 1$, then all entries in some upper right corner of the matrix Ω will have zero of order $> m - 2$ at the origin. Indeed, if $i \leq k$ and $j \geq k + 1$, then the (i, j) th matrix element of the Pfaffian matrix Ω is obtained by multiplying the corresponding element ω'_{ij} of Ω' by t^d , $d = \nu_i - \nu_j \geq \nu_k - \nu_{k+1} > m - 1$. Since Ω' is Fuchsian, its entries have at most first order pole, thus the order of zero of all ω'_{ij} with $i \leq k$ and $j \geq k + 1$ will be greater than $m - 2$.

On the other hand, since the form Ω is globally defined on the whole sphere, its entries are rational 1-forms. By assumptions, these forms have at most simple poles at no more than $m - 1$ other points of \mathbb{P}^1 . Thus the order of zero at the origin cannot be greater than $m - 2$, unless the form is identically zero (the difference between the total number of poles and zeros for any rational form is always equal to 2). This necessarily implies that $\omega_{ij} \equiv 0$ for all combinations of i, j such that $i \leq k$ and $j \geq k + 1$.

But the simultaneous occurrence of a corner of identical zeros as was described above, in the (rational, i.e., globally defined) Pfaffian matrix Ω means that the coordinate subspace $\{x_1 = \cdots = x_k = 0\}$ is invariant by the system, hence by all monodromy operators, contrary to the irreducibility assumption.

Thus for the case where the diagonal entries ν_i are arranged in the nonincreasing order, the difference between any two *consecutive* numbers cannot be greater than $m - 2$. Hence the difference between *any* two ν_i is no greater than $(m - 2)(n - 1)$ in the absolute value, and this assertion is already independent on the order of these numbers. \square

This lemma immediately implies an impossibility result of Riemann–Hilbert type. It provides for a wide class of holomorphic bundles on which the Riemann–Hilbert problem admits no solution.

Theorem 18.12. *An irreducible matrix group with m generators cannot be realized as a holonomy group of a meromorphic connexion with $m+1$ singular points on a holomorphic bundle with the splitting type $D = \{d_1, \dots, d_n\}$ over \mathbb{P} , unless the following inequalities hold,*

$$|d_i - d_j| \leq (m - 2)(n - 1) \quad \forall i, j = 1, \dots, n. \quad (18.7)$$

Proof. Assume that such a connexion ∇ exists and the point at infinity is singular for it.

Consider the meromorphic trivialization of the bundle π by the cochain (18.4) described in §18B. This trivialization does not change the holonomy group, thus the connexion ∇_0 on the trivial bundle is also irreducible.

A fundamental matrix solution for horizontal sections $\nabla_0 X = 0$ near infinity has the form $X(t) = t^{-D}G(t)Y(t)$, where G is holomorphically invertible at infinity and $Y(t)$ is a fundamental solution of the equation $\nabla Y = 0$ near infinity. This follows from the explicit form of the trivialization (18.4).

By assumption, ∇ is Fuchsian, so the logarithmic derivative $dY \cdot Y^{-1}$ of the matrix function $Y(t)$ has a first order pole. Since G is holomorphic and invertible, the logarithmic derivative of the product GY also has a first order pole at infinity. If one of the equalities (18.7) is violated, after change of the independent variable $t \mapsto 1/t$ which sends infinity to the origin, it would contradict Lemma 18.11, since all other singularities of ∇_0 are Fuchsian. \square

Remark 18.13. The assertion of Theorem 18.12 is remarkable for the following reason. In construction of the holomorphic bundle as in Theorem 18.4 each monodromy operator M_j can be realized by infinitely many different local connexion forms Ω_j . Even if only the Euler equations are used, still there is a freedom to choose matrix logarithms which can be used to produce infinitely many holomorphically nonequivalent types of singularities at each point $a_j \in \Sigma$. One could expect that combining these nonequivalent singularities and patching them together, one can produce infinitely many different splitting types of holomorphic bundles.

Theorem 18.12 claims that the *global condition of irreducibility of the monodromy group* imposes a global restriction that is compatible with *finitely many different splitting types* only. In the next subsection we will show that in fact one of these splitting types admits holomorphic trivialization.

18D. Bolibruch–Kostov theorem. The most remarkable positive result on solvability of the Riemann–Hilbert problem was discovered independently by A. Bolibruch [Bol92] and V. Kostov [Kos92].

Theorem 18.14. *Any irreducible matrix group can be realized as the holonomy group of a Fuchsian connexion on the trivial vector bundle over \mathbb{P} .*

In other words, any monodromy data $\{M_1, \dots, M_m\}$ such that the matrices M_j do not have a common nontrivial invariant subspace, can be realized by a linear system with rational matrix function Ω having only simple poles at the specified points and no other singularities. This is the strongest, the third form of the solvability of the Riemann–Hilbert problem on p. 312.

Proof. Unlike the previous demonstrations, when we started from an *arbitrary* Fuchsian connexion defined by a collection of connexion 1-forms $\{\Omega_\alpha\}$ on an abstract holomorphic vector bundle, realizing the specified holonomy

group, this time we will use explicitly the freedom in the choice of the connexion forms Ω_j realizing each holonomy. It is sufficient to vary only one of the forms.

More precisely, we will assume that one of the preassigned singularities is at the point $t = \infty$ and the corresponding holonomy operator is *upper-triangular*. Such a singularity can be realized by the local connexion form $\Omega_m = A_m t^{-1} dt$ with an upper-triangular residue matrix $A = A_m$ with a fundamental matrix solution t^A . Yet without changing the holonomy Δ_m we can replace Ω_m by a meromorphically gauge equivalent 1-form which corresponds to replacing the matrix solution by another function $t^N t^A$. More specifically, we consider the new connexion form of the structure

$$\Omega'_N = N t^{-1} dt + t^N \Omega_m t^{-N}. \quad (18.8)$$

The term $t^N \Omega_m t^{-N}$ has the first order at infinity by the usual arguments, if the matrix A is upper-triangular and the integer numbers ν_i follow in the ascending order, $\nu_1 < \dots < \nu_n$.

Denote by π_N the holomorphic vector bundle obtained by gluing together the connexions $\Omega_1, \dots, \Omega_{m-1}, \Omega_N$. This bundle carries the meromorphic connexion represented by the above cochain of 1-forms, which will be denoted ∇_N . The connexion ∇_N is irreducible by construction. Hence the splitting type $D = \text{diag}\{d_1, \dots, d_n\} = D(N)$ of the bundle itself is constrained by the inequalities from Theorem 18.12.

Consider the meromorphic trivialization (18.4) of the bundle π_N . As usual, it has only Fuchsian singularities at all finite points, and a regular singularity at infinity with a fundamental matrix solution of the form

$$X(t) = t^{-D} G_1(t) t^N t^A,$$

where the splitting diagonal matrix D and the holomorphic invertible matrix $G_1(t)$ depend on the diagonal matrix N . By the Permutation Lemma 16.36, there exists a monopole gauge transform that brings the fundamental solution X' into the form

$$X'(t) = G'(t) t^{-D'} t^N t^A = G'(t) t^{-D'+N} t^A, \quad G' \in \text{GL}(n, \mathcal{O}(\mathbb{P}, \infty)),$$

where the integer diagonal matrix D' has the same entries d_i but in a permuted order.

Yet (and this is the key step of the proof) if the sequence ν_i was ascending sufficiently fast and the sequence d'_i is constrained by the inequality $|d'_i - d'_{i+1}| \leq (m-2)(n-1)$ (cf. with (18.6)), then the sequence $\nu'_i = \nu_i - d'_i$ is also ascending (increasing). To ensure the monotonicity, it is sufficient to require that

$$\nu_{i+1} - \nu_i > (m-2)(n-1) \quad \text{for all } i = 1, \dots, n-1.$$

The monotonicity of ν'_i is sufficient to guarantee that the singularity with the fundamental solution $t^{-D'+N}t^A$ is Fuchsian (recall that A is upper-triangular). Left multiplication by the holomorphically invertible matrix G' does not change this fact: after meromorphic trivialization F and the subsequent monopole gauge transform we obtain the trivial bundle with a Fuchsian connexion on it. \square

18E. Bolibruch counterexample. In this section we describe a reducible matrix group that cannot be realized as the holonomy of a Fuchsian connexion on the trivial bundle. More precisely, we describe an obstruction that prevents a given matrix group to be realized by a Fuchsian connexion on the trivial bundle. A similar obstruction is obtained for nontrivial bundles.

Recall that each linear operator $M \in GL(n, \mathbb{C})$ over the field \mathbb{C} always has at least one invariant subspace of each dimension $k = 1, \dots, n-1$ (Exercise 18.6). There are operators for which there are no other invariant subspaces.

Definition 18.15. A linear operator $M: \mathbb{C}^n \rightarrow \mathbb{C}^n$ will be called a *monoblock*, if its Jordan normal form consists of a single block of maximal size.

By definition, the spectrum of each monoblock is a *singleton*, i.e., the operator has a single eigenvalue ν and for any $k \leq n$ the power $(M - \nu E)^k$ has the rank *exactly* equal to $n - k$.

Lemma 18.16. *A monoblock operator on a complex n -space has exactly one invariant subspace of each intermediate dimension k between 1 and $n-1$. In a basis in which M has an upper-triangular matrix, this subspace is spanned by the first k vectors.*

Proof. Without loss of generality assume that the unique eigenvalue of M is zero, $\nu = 0$, that is, M is nilpotent.

If V is an invariant subspace of dimension $k \leq n$ for M , then the restriction of M on V must also be nilpotent, more precisely, $M^k|_V = 0$. But for a nilpotent operator of class B the rank of M^k is exactly $n - k$, which means that $\dim \text{Ker } M^k = k$, and hence V must coincide with $\text{Ker } M^k$, being thus uniquely defined.

It remains to notice that for an upper-triangular nilpotent matrix M , $\text{Ker } M^k$ consists of the first k basic vectors. \square

Monoblocks are rather rigid; for instance, any monoblock admits a unique matrix logarithm modulo a scalar matrix (cf. with Remark 18.13) which is also a monoblock (Problem 18.7). In other words, if a monoblock is realized as a holonomy of a Fuchsian singular point which is linearizable

(i.e., equivalent to an Euler system), then the corresponding residue matrix is a monoblock as well.

In the class of non-Euler systems one may have really different (not locally holomorphically equivalent) Fuchsian realizations of a monoblock holonomy. In particular, a Fuchsian singular point with a monoblock holonomy can have different (though necessarily, resonant) eigenvalues. The following assertion may be considered as a true “nonlinear” analog of the fact that a monoblock matrix has a monoblock logarithm.

Lemma 18.17. *If a Fuchsian singular point of a connexion ∇ of rank n has a monoblock local monodromy, then for each intermediate dimension k , $1 \leq k \leq n - 1$, there exists exactly one holomorphic subbundle $\pi_k: L_k \rightarrow (\mathbb{C}, 0)$, $\text{rank } \pi_k = k$, invariant by ∇ , and the residue of the restriction $\nabla_k = \nabla|_{L_k}$ of the connexion on the subbundle satisfies the inequalities*

$$\frac{1}{k} \text{tr } \text{res}_0 \nabla_k \geq \frac{1}{n} \text{tr } \text{res}_0 \nabla. \quad (18.9)$$

The equality is possible only for all values $k = 1, \dots, n - 1$ simultaneously and only in the case where the residue matrix $\text{res}_0 \nabla$ has a single eigenvalue.

Proof. The assertion is purely local, so it can be verified for a linear system in the Poincaré–Dulac–Levelt normal form (16.7).

Since the monodromy has a single eigenvalue, all eigenvalues $\lambda_1, \dots, \lambda_n$ of the residue $\text{res}_0 \nabla$ fall in the same resonant group, i.e., differ only by integers, as follows from the explicit formula (16.11). Arranging the eigenvalues in the nonincreasing order $\lambda_1 \geq \dots \geq \lambda_n$ (recall again that this means non-negativity of all differences $\lambda_i - \lambda_j \geq 0$ for $i < j$). In these settings the connexion matrix $A(t)$ in (16.7) is upper-triangular (Remark 16.14).

For the system (16.7) in the upper-triangular form, each coordinate subspace $L_k = \{x_{k+1} = \dots = x_n = 0\} \subset (\mathbb{C}, 0) \times \mathbb{C}^n$ generated by the first k coordinate vectors, is invariant and hence constitutes a “constant” invariant subbundle $\pi_k: L_k \rightarrow (\mathbb{C}, 0)$ of rank k . Moreover, the trace of the residue matrix restricted on L_k is the sum of the *first* k eigenvalues $\lambda_1, \dots, \lambda_k$ of the residue matrix. Yet since the *largest eigenvalues come first*, we instantly obtain the inequalities

$$\frac{1}{k} \text{tr } \text{res}_0 \nabla_k \geq \frac{1}{n} \text{tr } \text{res}_0 \nabla$$

for the restrictions ∇_k of ∇ on the subbundles L_k . The equality is possible if and only if the smallest and the largest eigenvalues are equal, i.e., if $\lambda_1 = \dots = \lambda_n = \lambda$.

To prove the uniqueness, note that since the connexion is of Bolibruch type, each invariant subspace $\ell_k \in \mathbb{C}^n$ of rank k for the monodromy operator $M \in \text{GL}(n, \mathbb{C})$ is unique and extends as a holomorphic invariant subbundle, necessarily coinciding with L_k . \square

Globalization of this construction leads to a very important notion which will play a central role in the construction of the counterexample.

Definition 18.18. A meromorphic connexion on a holomorphic vector bundle is called a *Bolibruch connexion*, if it has a nontrivial invariant subbundle, all the singular points of the connection are Fuchsian, and the local holonomy of each singular point is a monoblock operator.

The global analog of Lemma 18.17 then takes the following form. Note that, unlike the “inequality” between the complex numbers, understood in the “artificial” sense (11.3), the inequality (18.10) relates two rational numbers.

Theorem 18.19. *Suppose that a Bolibruch connexion ∇' on a holomorphic bundle π' over \mathbb{P} has a nontrivial invariant subbundle π . Then the ratio of degree to rank for the subbundle is greater or equal to this ratio for the ambient bundle,*

$$\pi \subseteq \pi' \implies \frac{\deg \pi}{\text{rank } \pi} \geq \frac{\deg \pi'}{\text{rank } \pi'}. \quad (18.10)$$

The equality occurs if and only if the spectrum of each singularity of ∇' is a singleton.

Proof. Let $\nabla = \nabla'|_{\pi}$ be the restriction of ∇' on the subbundle $\pi: L \rightarrow \mathbb{P}$. Denote $k = \text{rank } \pi$, $n = \text{rank } \pi'$. By Corollary 17.35, the degree of both bundles is equal to the sum of traces of the residues of all singularities.

Adding together the local inequalities (18.9) over all singularities $a \in \Sigma$, we conclude that

$$\frac{1}{k} \deg \pi = \sum \frac{1}{k} \text{res}_a \text{tr } \nabla \geq \sum \frac{1}{n} \text{res}_a \text{tr } \nabla' = \frac{1}{n} \deg \pi'.$$

The equality occurs if and only if all spectra are singletons. \square

Together with Corollary 17.25, Theorem 18.19 imposes rather strong restrictions on Bolibruch connexions on the trivial bundle.

Theorem 18.20. *For a Bolibruch connexion on a trivial bundle, the spectra of all singularities must necessarily be singletons, and the invariant subbundle itself must be trivial.*

Proof. If π_0 is a trivial bundle and π its subbundle invariant by a Bolibruch connexion ∇ , then $\deg \pi_0 = 0$. By Theorem 18.19, we have $\deg \pi \geq 0$ and by Corollary 17.25, $\deg \pi \leq 0$. Together these inequalities leave only one possibility $\deg \pi = 0$, so that in both assertions the extreme cases occur. This implies both assertions. \square

We have arrived at the main step of the impossibility proof. The assumptions of Theorem 18.20 (reducibility and Jordan block structure of the monodromy matrices) are imposed on the *holonomy group* of the connexion ∇ rather than on the *connexion itself*. However, the assertion concerns the connexion (more specifically, its residue matrices). In other words, Theorem 18.20 implicitly describes an obstruction to realizability of a reducible monodromy data of monoblock operators by a Fuchsian connexion on the trivial bundle. In particular, we arrive at the following result which is just a geometric reformulation of Theorem 16.33.

Theorem 18.21. *If a Fuchsian connexion ∇ on a holomorphic bundle π of rank 4 over \mathbb{P} with three singular points has the monodromy matrices*

$$\begin{pmatrix} 1 & 1 & & \\ & 1 & 1 & \\ & & 1 & 1 \\ & & & 1 \end{pmatrix}, \begin{pmatrix} 3 & 1 & 1 & -1 \\ -4 & -1 & 1 & 2 \\ & & 3 & 1 \\ & & -4 & -1 \end{pmatrix}, \begin{pmatrix} -1 & & 2 & -1 \\ & 4 & -1 & 1 \\ & & -1 & \\ & & & 4 & -1 \end{pmatrix}, \quad (18.11)$$

then the bundle π is necessarily nontrivial.

Proof. One can easily see that all three matrices (18.11) are monoblocks (with the respective eigenvalues $\mu_{1,2} = 1$, $\mu_3 = -1$) and have an invariant subspace spanned by the first two coordinates. Hence the connexion ∇ realizing the corresponding monodromy data, is necessarily a Bolibruch connexion. If the bundle π were trivial then by Theorem 18.20 each residue matrix $A_j = \text{res}_{a_j} \nabla$ must have a singleton spectrum λ_j such that $\exp 2\pi i \lambda_j = \mu_j$ for all $j = 1, 2, 3$. Resolving the corresponding equations $\exp 2\pi i \lambda_{1,2} = 1$, $\exp 2\pi i \lambda_3 = -1$, we obtain the congruences

$$\lambda_1 \equiv \lambda_2 \equiv 0 \pmod{\mathbb{Z}}, \quad \lambda_3 \equiv \frac{1}{2} \pmod{\mathbb{Z}}. \quad (18.12)$$

On the other hand, by the index theorem (Corollary 17.35) we would have for the trivial bundle the equality

$$\deg \pi = 0 = \text{tr } A_1 + \text{tr } A_2 + \text{tr } A_3 = 4(\lambda_1 + \lambda_2 + \lambda_3).$$

The resulting impossible congruence $0 \equiv 2 \pmod{4\mathbb{Z}}$ proves that the bundle π cannot be trivial. \square

This argument gives an alternative proof of Theorem 16.33.

Exercises and Problems for §18.

Problem 18.1. Prove that the Riemann–Hilbert problem is solvable if all monodromy matrices commute, $[M_i, M_j] = 0$ for all i, j .

Exercise 18.2. Write a detailed proof of Theorem 18.5.

Exercise 18.3. Prove that the Riemann–Hilbert problem can be always solved by a Fuchsian linear system for any monodromy data if the meromorphic matrix form is allowed to have a single extra singular point with identical holonomy at any preassigned point off the singular locus Σ .

Problem 18.4. Construct an example of an irregular singularity and a subspace invariant by the (local) monodromy, which does not extend as an invariant holomorphic subbundle over a neighborhood of the singular point (cf. with Proposition 18.8).

Problem 18.5. Prove that any meromorphic rectangular matrix function $X(t)$ of size $n \times k$, $k < n$ can be locally near $t \in (\mathbb{C}, 0)$ represented under the form $X(t) = L(t)D(t)R(t)$, where $L(t)$ and $R(t)$ are holomorphic invertible square matrices of sizes $n \times n$ and $k \times k$ respectively, and $D(t)$ is the rectangular truncation (first k columns) of a diagonal matrix which has only integer powers t^{ν_i} or zeros on the diagonal.

Exercise 18.6. Prove that any operator $M \in \text{GL}(n, \mathbb{C})$ has at least one invariant subspace $L_k \subset \mathbb{C}^n$ of each intermediate dimension $k = 1, \dots, n - 1$.

Problem 18.7. Prove that any two matrix logarithms A, A' of the same monoblock operator differ by an integer multiple of the identity matrix modulo conjugacy:

$$\exp A = \exp A' \text{ is a monoblock} \implies A - CA'C^{-1} = 2\pi ikE$$

for a suitable integer number $k \in \mathbb{Z}$ and an invertible conjugacy matrix $C \in \text{GL}(n, \mathbb{C})$. Prove that each logarithm is also a monoblock.

Problem 18.8. Prove that the Riemann–Hilbert problem is always solvable in the classical sense (i.e., on the trivial bundle) in dimension 2.

Problem 18.9. Prove that the monodromy data with one diagonal matrix can be realized by infinitely many nonequivalent Fuchsian systems.

Problem 18.10. Prove that any irreducible monodromy data can be realized by infinitely many nonequivalent Fuchsian systems.

Problem 18.11. Prove that the Riemann–Hilbert problem is nonsolvable in all dimensions greater than 4.

Problem 18.12. Prove the following generalization of Theorem 18.12. Let ∇ be a meromorphic *non-Fuchsian* connexion on a holomorphic vector bundle of rank n and the splitting type $D = \{d_1, \dots, d_n\}$ with at least one Fuchsian singularity. Denote by m the total order of poles of all singularities. Prove that if for some pair of indices $|d_i - d_j| \geq (m - 2)(n - 1)$, then the connexion ∇ is reducible, i.e., has an invariant subbundle.

19. Linear n th order differential equations

Linear high order scalar differential equation can be reduced to a rather special class of *companion* linear systems which are naturally defined connexions on the *jet bundle*. Because of the special form, regular singular