Problem 16.6. Let $\Delta_a\colon \tau_a \to \tau_a$ be the holonomy operator corresponding to a simple positive loop around the origin beginning and ending at a nonsingular point $a \neq 0$ for a Fuchsian system $t\dot{x} = (A_0 + tA_1 + \cdots)x$. Prove that Δ_a depends analytically on a as $a \neq 0$, extends (as an analytic matrix function) at the origin $a = 0$ and the limit Δ_0 is equal to $\exp 2\pi i A_0$. Show that the operators Δ_a are conjugate to each other for all $a \neq 0$, but not necessarily to Δ_0.

Problem 16.7. Bring to the Poincaré–Dulac–Levelt normal form the linear systems with the matrix 1-form $\Omega = A(t)\frac{dt}{t}$, where $A(t)$ is one of the following matrix functions,

$$\begin{pmatrix} 1 & \sin 2t \\ & 2 \end{pmatrix}, \qquad \begin{pmatrix} 1 & e^t - 1 & t^3 \\ & 2 & t^2 \\ & & 3 \end{pmatrix}.$$

Problem 16.8. Prove that for any resonant tuple of the form $\boldsymbol{\lambda}^1 = (\lambda, \lambda + k)$ or $\boldsymbol{\lambda}^2 = (\lambda, \lambda + k, \lambda + k + m)$ there exists but a finite number of normal forms of equations with a Fuchsian singular point, for which the residue matrix has the spectrum $\boldsymbol{\lambda}^1$ or $\boldsymbol{\lambda}^2$.

17. Global theory of linear systems: holomorphic vector bundles and meromorphic connexions

Linear systems appear in a natural way by *linearization* of arbitrary complex one-dimensional holomorphic foliations along particular leaves (usually, separatrices). Example of such linearization for foliations on complex surfaces already appeared in the computation of the vanishing holonomy group in §10**D** and in slightly more general context in §14**B**. Both these examples suggest that, while locally a linear system "lives" on cylinders which are Cartesian products of the base leaf L by a complex linear space of the complementary dimension, globally the situation may be nontrivial. In particular, it may be impossible to define the linearized system globally over L by a single meromorphic 1-form (matrix or even scalar): the nontrivial relationship between 1-forms θ_1 and ϑ_1 in (10.9) shows that the linearized system is defined on a more general object than the "simple" Cartesian product $\mathbb{E} \times \mathbb{C}$. This object is called (holomorphic) *vector bundle.*

The material exposed in this section is rather standard and can be found in numerous sources, of which we recommend the books [**For91**, §29, §30] and [**Bol00**], but also [**GH78**, §0.5] and [**Wel80**, §2].

17A. Holomorphic vector bundles. A real or complex vector bundle of rank n over a topological manifold T (the "horizontal" base) is a topological manifold which is "built" from Cartesian cylinders $U_\alpha \times \mathbb{R}^n$ or $U_\alpha \times \mathbb{C}^n$ respectively, where U_α is a chart on T, in the same way as the base manifold is built from the charts U_α themselves. The added value is the linear structure along the "vertical" fibers $\{a\} \times \mathbb{R}^n$, resp., $\{a\} \times \mathbb{C}^n$. We will be interested only in the complex case. The formal definition looks as follows.

Definition 17.1. Let $\pi\colon S \to T$ be a continuous map between two topological spaces. A map Φ is called a *local trivialization* (sometimes *trivializing chart*, or simply *trivialization*) of π over an open subset $U \subseteq T$, if $\Phi\colon \pi^{-1}(U) \to U \times \mathbb{C}^n$ is a homeomorphism which conjugates π with the projection of the Cartesian product (cylinder) $\pi_0\colon U \times \mathbb{C}^n \to U$ on the first component, so that $\pi_0 \circ \Phi = \pi$.

Trivializations play the role of special coordinate charts keeping track of the linear structure on the fibers.

Definition 17.2. The topological space S together with a continuous map (projection) $\pi\colon S \to T$ is called a *topological complex vector bundle* or rank n over a topological space T (called the *base*), if:

(1) for any point $a \in T$ of the base there exists an open neighborhood $U_\alpha \ni a$ and a trivialization Φ_α of π over U_α,

(2) the family of trivializations $\{\Phi_\alpha\}$ respects the linear structure of the *fibers* $\pi^{-1}(a)$: if Φ_α, Φ_β are two trivializations of π over two open domains with the nonempty intersection $U_{\alpha\beta} = U_\alpha \cap U_\beta$, then the *transition map* between them is a gauge transform fibered over the identity map as in §15**D**, i.e.,

$$\begin{gathered} \Phi_\beta \circ \Phi_\alpha^{-1}\colon U_{\alpha\beta} \times \mathbb{C}^n \to U_{\alpha\beta} \times \mathbb{C}^n, \\ \Phi_\beta \circ \Phi_\alpha^{-1}(a, x) = \bigl(a, H_{\beta\alpha}(a)\, x\bigr), \qquad H_{\beta\alpha}(a) \in \mathrm{GL}(n, \mathbb{C}),\ a \in U_{\alpha\beta}. \end{gathered} \tag{17.1}$$

The triplet $\pi\colon S \to T$ is called a *holomorphic* complex vector bundle, if both S and T are holomorphic manifolds, π is a holomorphic projection which admits *biholomorphic* trivialization near each point of T. In this case the transition maps are biholomorphic gauge transformations.

Preimages of points $\tau_a = \pi^{-1}(a)$ are called *fibers* of the vector bundle. The space S is called the *total space* of the vector bundle.

The bundles will usually be denoted by the symbols for the corresponding projections, provided that the two other components of the triplet (the total space and bundle) are clearly defined by the context.

Geometry provides a vast source of bundles. For any holomorphic manifold M of complex dimension n the collection of tangent vectors attached to different points of M has a natural structure of a holomorphic vector bundle of rank n over the base M, called the *tangent bundle*. Indeed, if $U \subset \mathbb{C}^n$ is a domain in the affine space, then vectors tangent to different points of U can be identified with elements of the vector space $\mathbb{C}^n$ itself. Thus every chart on M, defined in a domain $U \subset M$ provides a local trivialization of the tangent bundle. The tangent bundle is usually denoted $\mathbf{T}M$. In a similar way the cotangent bundle $\mathbf{T}^*M$ is defined as the collection of covectors (linear functionals on tangent spaces) at all points of M (see Problem 17.1).

17B. Cocycles. Obviously, if $\pi\colon S \to T$ is a topological (resp., holomorphic) vector bundle, then for each two local trivializations over overlapping domains the matrix functions

$$H_{\beta\alpha}\colon U_{\alpha\beta} \to \mathrm{GL}(n,\mathbb{C}), \qquad U_{\alpha\beta} = U_\alpha \cap U_\beta, \tag{17.2}$$

is continuous (resp., holomorphic) together with its inverse $H_{\beta\alpha}^{-1}$. Since the construction is symmetric with respect to the two trivializations, this inverse is the transition matrix $H_{\alpha\beta}$, i.e., we have the identities

$$H_{\alpha\beta} \cdot H_{\beta\alpha} \equiv E \qquad \text{on} \quad U_{\alpha\beta}. \tag{17.3}$$

Besides, if U_α, U_β and U_γ are *three* domains with the pairwise intersections $U_{\alpha\beta}, U_{\beta\gamma}, U_{\alpha\gamma}$ and a nonvoid triple intersection $U_{\alpha\beta\gamma}$, then

$$H_{\alpha\beta} \cdot H_{\beta\gamma} \cdot H_{\gamma\alpha} \equiv E \qquad \text{on} \quad U_{\alpha\beta\gamma}. \tag{17.4}$$

Indeed, this composition corresponds to the transition between the trivializations Φ_α, Φ_γ and Φ_β (in the specified order) back to Φ_α.

Definition 17.3. Let $\mathfrak{U} = \{U_\alpha\}$ be an open covering of the base T. A *holomorphic matrix cocycle* inscribed in this covering (or subordinated to this covering) is a collection of holomorphic matrix functions $\mathcal{H} = \{H_{\alpha\beta}\}$ defined in all nonempty pairwise intersections $U_{\alpha\beta}$ and satisfying the identities (17.3) and (17.4) on all nonempty double (resp., triple) intersections.

Definition 17.4. A *holomorphic matrix cochain* $\mathcal{G}$ subordinated to the covering $\mathfrak{U}$, is a collection of holomorphic matrix functions $G_\alpha \in \mathrm{Mat}(n, U_\alpha)$ defined and holomorphic in the domains of the covering. In a similar way meromorphic, vector and other types of cochains are defined with obvious modifications.[2]

Definition 17.5. The operator transforming a cochain $\mathcal{G} = \{G_\alpha\}$ into the cocycle $\mathcal{H} = \{H_{\alpha\beta}\}$ with $H_{\alpha\beta} = G_\alpha G_\beta^{-1}$, is called the *coboundary* (or *multiplicative matrix coboundary*, if necessary to distinguish it from similar operators).

Any family of trivializations of a holomorphic vector bundle defines a holomorphic matrix cocycle. Conversely, any holomorphic matrix cocycle inscribed in an open covering of T determines a holomorphic vector bundle over T.

[2]The notions of cocycle and cochain belong to algebraic topology which defines cohomology with coefficients in different sheaves. It would be more appropriate to use the terms 1-cocycle and 0-cochain rather than simply cocycle and cochain, yet we will never need the general case of k-cochains or k-cocycles in this book.

Theorem 17.6. *Any matrix cocycle inscribed in a covering of a holomorphic manifold T, can be realized as the collection of transition gauge maps between local trivializations of a suitable holomorphic vector bundle over the base T.*

Proof. Consider the disjoint union of cylinders $\widetilde{S} = \bigsqcup_\alpha U_\alpha \times \mathbb{C}^n$ together with the equivalence relation on it, identifying the points

$$U_\alpha \times \mathbb{C}^n \ni (a,x) \sim (a',x') \in U_\beta \times \mathbb{C}^n \iff a = a' \in U_{\alpha\beta} \text{ and } x' = H_{\beta\alpha}x.$$

This relation is indeed symmetric by (17.3) and transitive by (17.4). The quotient space $S = \widetilde{S}/\sim$ by this relation obtains thus a natural structure of a holomorphic manifold with the charts $U_\alpha \times \mathbb{C}^n$. The Cartesian projections $\pi_\alpha \colon U_\alpha \times \mathbb{C}^n \to U_\alpha$ respect the equivalence and hence together define an analytic map $\pi\colon S \to T$. The cylinders $U_\alpha \times \mathbb{C}^n$ provide trivializations of the map π over U_α, and the transition maps between these trivializations tautologically coincide with the gauge transforms defined by the specified matrix functions from the cocycle. □

Description of vector bundles by matrix cocycles provides analytic tools (methods of theory of analytic matrix functions) for working in the geometric category of vector bundles.

Example 17.7. The trivial vector bundle $\pi\colon T \times \mathbb{C}^n \to T$, $\pi(a,x) = a$, of any rank n exists over any base T and is associated with the trivial cocycle $\{H_{\alpha\beta} = E\}$ inscribed in an arbitrary covering of T.

The definition of a vector bundle does not specify any particular choice of the trivializations (there mere existence is required). Clearly, if Φ_α is a trivialization of a vector bundle π over a domain $U_\alpha \subseteq T$ and $G_\alpha \colon U_\alpha \times \mathbb{C}^n \to U_\alpha \times \mathbb{C}^n$ a collection of invertible gauge map fibered over the identity, then $\Phi'_\alpha = G_\alpha \circ \Phi_\alpha$ is another trivialization over the same domain U_α. The cocycle $\mathcal{H}' = \{H'_{\alpha\beta}\}$ of the transition maps associated with the new collection of trivializations, is related to the initial cocycle as follows:

$$H'_{\alpha\beta} G_\beta = G_\alpha H_{\alpha\beta} \qquad \text{on} \quad U_{\alpha\beta}. \tag{17.5}$$

Definition 17.8. Two cocycles $\mathcal{H} = \{H_{\alpha\beta}\}$ and $\mathcal{H}' = \{H'_{\alpha\beta}\}$ inscribed in the same covering $\mathfrak{U} = \{U_\alpha\}$ are *equivalent*, if there exists a *holomorphic matrix cochain* $\mathcal{G} = \{G_\alpha\}$, such that (17.5) holds.

Summarizing, we conclude that each holomorphic vector bundle over the base T is associated with a family of equivalent holomorphic matrix cocycle inscribed in some open covering $\mathfrak{U} = \{U_\alpha\}$ of T. Conversely, any matrix cocycle can be realized by a suitable bundle.

The question that was not yet addressed is equivalence of bundles obtained from *different* coverings. Clearly, if a covering $\mathfrak{U} = \{U_\alpha\}$ is a refinement of another, more coarse covering $\mathfrak{U}' = \{U'_i\}$, i.e., if each U_α entirely belongs to one of the larger domains U'_i, then any cocycle inscribed in $\mathfrak{U}'$ can be refined to a cocycle inscribed in $\mathfrak{U}$, by restriction (postulating the identical transitions $H_{\alpha\beta} = \mathrm{id}$, if both U_α and U_β belong to the same larger domain U'_i). This allows us to define equivalence of two cocycles $\mathcal{H}, \mathcal{H}'$ inscribed in two different coverings $\mathfrak{U}, \mathfrak{U}'$, by passing to cocycles inscribed in the common refinement $\mathfrak{U}'' = \{U_\alpha \cap U'_i\}$.

Replacing the domains U_α by smaller ones, we can (and will always) assume that each of them are topological disks with smooth boundaries.

A difficult problem is to pass from fine to more coarse coverings. To that end one has to combine two trivializations over overlapping domains U_α, U_β into a trivialization over the union $U_\alpha \cup U_\beta$. This problem will be discussed in detail later, in §17**J**.

17C. Operations on bundles. Speaking informally, a holomorphic bundle is a union of linear spaces (fibers) parameterized by points of the base T in a locally trivial way. Most constructions of linear algebra can be translated into the category of vector bundles by implementing these constructions "fiberwise". We provide a brief glossary of the most basic terms.

Definition 17.9. A (holomorphic) *bundle map between two vector bundles* $\pi\colon S \to T$ and $\pi'\colon S' \to T'$ is a holomorphic map $F\colon S \to S'$ between the total spaces, which maps fibers of π linearly to fibers of π'.

Formally this means that there exists a map $f\colon T \to T'$ between the bases, such that $\pi' \circ F = f \circ \pi$. We say that the map F is *fibered over* f. Two vector bundles are equivalent, if there exists an invertible holomorphic bundle map between them.

To write bundle maps "in coordinates" we need to choose a pair of trivializations near a given point $a \in T$ and its image $a' = f(a)$. Consider a pair of domains $U_\alpha \subset T$ and $U'_i \subset T'$, containing a and a' respectively, and let Φ_α, Φ'_i respectively be two collections of trivializations of these two bundles. Then a bundle map becomes a gauge map between $U_\alpha \times \mathbb{C}^n$ and $U'_i \times \mathbb{C}^m$ (we do not assume that the two bundles have the same rank). In other words, in the trivializing charts the map $\Phi'_i \circ F \circ \Phi_\alpha^{-1}$ takes the form

$$U_\alpha \times \mathbb{C}^n \to U'_i \times \mathbb{C}^m, \qquad (a, x) \mapsto \bigl(f(a), F_{\alpha,i}(a) \cdot x\bigr),$$

with a $(n \times m)$-holomorphic matrix function $F_{\alpha,i}$. If instead of the trivialization Φ_α another trivialization Φ_β of the total space at the source is chosen, the matrix function $F_{\alpha,i}$ will be replaced by the matrix function $F_{\beta,i}$ which on the intersection $U_{\alpha\beta}$ satisfies the identity $F_{\beta,i}(a) = F_{\alpha,i}(a) \cdot H_{\alpha\beta}(a)$. A similar rule applies when changing the trivialization of the target total space.

Example 17.10. If the bundle S' is trivial (of some dimension m), then a bundle map between S and S' is defined by a cochain $\mathcal{F} = \{F_\alpha\}$ such that $F_\alpha \cdot H_{\alpha\beta} = F_\beta$.

Conversely, a map from the trivial bundle S' to S is defined by a cochain $\mathcal{G} = \{G_\alpha\}$ such that $H_{\alpha\beta} \cdot G_\beta = G_\alpha$.

Definition 17.11. A holomorphic cocycle $\mathcal{H} = \{H_{\alpha\beta}\}$ is called *solvable*, if there exists a holomorphic matrix cochain $\mathcal{G} = \{G_\alpha\}$ such that

$$H_{\alpha\beta} = G_\alpha G_\beta^{-1}. \tag{17.6}$$

By this definition, solvable cochains correspond to bundles which are holomorphically equivalent to the trivial bundle. In analytic terms cocycle is solvable if and only if it is equivalent to the trivial cocycle.

The general construction of a bundle map becomes more transparent if both the source and the tangent bundle π, π' are over the same base and the map is fibered over the identity. In this case it is natural to use trivializations $\Phi_\alpha, \Phi'_\alpha$ inscribed in the same covering. In each pair of trivializations the map $F\colon S \to S'$ is associated with a holomorphic matrix function

$$\Phi'_\alpha \circ F \circ \Phi_\alpha^{-1}\colon U_\alpha \times \mathbb{C}^n \to U_\alpha \times \mathbb{C}^n, \qquad (a, x) \mapsto \bigl(a, F_\alpha(a)x\bigr).$$

In other words, a bundle map is associated with a holomorphic matrix cochain (the matrices can be nonsquare, if the ranks of π, π' are different.

On the overlapping of two domains the two matrix functions F_α, F_β are related by the identity

$$F_\beta = H'_{\beta\alpha} F_\alpha H_{\alpha\beta}, \qquad \text{i.e.,} \qquad H'_{\alpha\beta} F_\beta = F_\alpha H_{\alpha\beta} \qquad \text{on} \quad U_{\alpha\beta}, \tag{17.7}$$

where $\{H_{\alpha\beta}\}, \{H'_{\alpha\beta}\}$ are two cocycles defining the bundles π, π' respectively. This identity coincides with (17.5) if the matrices F_α are holomorphic invertible, which again illustrates the notion of equivalence of cocycles as equivalence of the corresponding bundles.

Other linear algebraic constructions are introduced in a similar way. A *subbundle* S' of a holomorphic bundle $\pi\colon S \to T$ is a holomorphic submanifold $S' \subseteq S$ such that the restriction of π on S' is itself a vector bundle of some rank k less or equal to the rank of S. If S' is a subbundle, then one can define the *quotient bundle* S/S', whose fibers are quotient spaces τ_a/τ'_a, $\tau_a = \pi^{-1}(a)$, $\tau'_a = \tau_a \cap S' \subseteq \tau_a$. Given any two bundles π, π' over the same base, one can construct their direct sum $\pi \oplus \pi'$, the tensor product $\pi \otimes \pi'$, dual bundle π^*, *etc.*

For instance, the tangent and cotangent bundles $\pi = \mathbf{T}M$ and $\pi^* = \mathbf{T}^*M$ over any holomorphic manifold M are dual to each other: for every point $a \in M$ there is a bilinear pairing $\pi^{-1}(a) \times {\pi^*}^{-1}(a) \to \mathbb{C}$ between the fibers of these bundles.

The notions of holomorphic vector bundle, cocycle, cochain make perfect sense in the case of minimal rank $n = 1$. This case is especially important, since 1×1-matrices commute, and hence it is much easier to study cocycles and equivalence. To distinguish this case, bundles of rank 1 are called *line bundles*.

One construction very important for future applications, allows us to associate with a vector bundle of any rank a line bundle called *determinant*, though a more appropriate name would be the maximal wedge product.

Note that for any linear space of dimension n its n-times wedge power (the wedge product of n copies of the space) is one-dimensional. Thus for any bundle π of rank n the wedge product

$$\det \pi = \underbrace{\pi \wedge \cdots \wedge \pi}_{n \text{ times}}$$

is a line bundle. Every linear map $H \in \mathrm{GL}(n, \mathbb{C})$ induces a map $\det H \in \mathrm{GL}(1, \mathbb{C})$ between the wedge products,

$$x_1 \wedge \cdots \wedge x_n \mapsto Hx_1 \wedge \cdots \wedge Hx_n = (\det H) \cdot x_1 \wedge \cdots \wedge x_n.$$

This allows us to define the determinant of a bundle in terms of cocycles.

Definition 17.12. The *determinant* of a vector bundle $\pi \colon S \to T$ of rank n, associated with a cocycle $\mathcal{H}$, is the holomorphic vector bundle of rank 1, associated with the cocycle

$$\det \mathcal{H} = \{h_{\alpha\beta}\}, \qquad h_{\alpha\beta} = \det H_{\alpha\beta}. \tag{17.8}$$

One can instantly verify that $\det \mathcal{H}$ is indeed a (scalar) cocycle. From (17.5) it follows that equivalent cocycles produce *the same* determinant cocycle.

17D. Classification of line bundles over the Riemann sphere. As a first step towards classification of holomorphic vector bundles of arbitrary rank over the Riemann sphere $\mathbb{P}$ in §17**J**, we now give a complete classification of line bundles over $\mathbb{P}$.

Consider the *standard covering* of the Riemann sphere $\mathbb{P}$ by an atlas of two charts, $U_0 = \{|t| < r_0\} \subseteq \mathbb{C}$ (the disk in the affine part with the chart t inherited from the affine line) and $U_1 = \{|t| > r_1\} \cup \{\infty\}$ with the chart $z = 1/t$, in which it also becomes an open disk. The intersection $A = U_{01}$ of these two charts is the circular annulus $A = \{r_1 < t < r_0\}$. The exact choice of the parameters $r_1 < r_0$ is not important.

A (holomorphic matrix) cocycle inscribed in the standard covering consists of a single pair of matrix functions $H_{01}(t) = H_{10}^{-1}(t)$ holomorphic and invertible in the annulus A. Such cocycles will be called *Birkhoff–Grothendieck* *cocycles*. For instance, a Birkhoff–Grothendieck cocycle of rank 1 is just a nonvanishing function $h(t) = h_{01}(t) = 1/h_{10}(t)$ holomorphic in the annulus.

Denote by ξ_d the line bundle corresponding to the *standard Birkhoff–Grothendieck cocycle*

$$\mathcal{L}_d = \{h_{01}, h_{10}\}, \quad h_{01}(t) = t^d\big|_{t\in A} = 1/h_{10}(t), \qquad d \in \mathbb{Z}, \tag{17.9}$$

in an annulus A. The integer number d will be referred to as the *degree* of the line bundle ξ_d and the corresponding standard cocycle.

Proposition 17.13. *Any scalar Birkhoff–Grothendieck cocycle $\mathcal{L} = \{h_{01}(t), h_{10}(t)\}$ is equivalent to one of the standard cocycles* (17.9) *of some degree d. Standard cocycles of different degrees are not equivalent to each other.*

To prove the proposition, we need an additive (rather than multiplicative) analog of holomorphic solvability of cocycles.

Lemma 17.14. *Let $U, U' \subseteq \mathbb{P}$ be two domains such that both of them and their intersection $V = U \cap U'$ have piecewise-smooth boundary.*

Then any function $v \in \mathcal{A}(V)$ holomorphic in V and continuous on the closure $\overline{V} = V \cup \partial V$ can be represented as the difference, $v = u - u'$, with $u \in \mathcal{A}(U)$, $u' \in \mathcal{A}(U')$.

For continuous functions the corresponding claim is obvious: among other solutions, one can simply choose $u' = 0$ (such a function is defined everywhere) and construct u as an arbitrary continuation of the function v from a closed subset $\overline{V}$ to a larger set U. However, holomorphic functions are very rigid, and Lemma 17.14 is a nontrivial (though simple) fact.

Proof of Lemma 17.14. The function v can be represented by the Cauchy integral over the boundary ∂V. This boundary can be represented as the *disjoint* union of two parts, $\partial V = B \sqcup B'$, with $B \subset \partial U$ and $B' \subset \partial U'$. Thus we have

$$v(t) = \frac{1}{2\pi i}\oint_{\partial V} \frac{f(z)\,dz}{z-t} = \frac{1}{2\pi i}\oint_{B} \frac{f(z)\,dz}{z-t} - \frac{1}{2\pi i}\oint_{-B'} \frac{f(z)\,dz}{z-t}.$$

The integral over B (resp., B') is holomorphic in $U \subset \mathbb{P} \smallsetminus B$ (resp., $U' \subset \mathbb{P} \smallsetminus B'$), and both are continuous on the boundary. □

Example 17.15. The function u holomorphic in the annulus $A = U_{01}$ as above, can be expanded in a *converging* Laurent series. Collecting together nonnegative and negative powers of t, we obtain two series converging in the respective disks $U_0, U_1 \subset \mathbb{P}$.

Proof of Proposition 17.13. There exists a unique integer number d such that the argument of the function $t^{-d}h(t)$ is a well-defined function in the annulus $A = U_{01}$. This number is equal to the index (rotation number) of the loop $h(\mathbb{S}^1)$ around the origin, where $\mathbb{S}^1$ is the unit circle $\{|t| = 1\}$.

For such a choice of d the function $t^{-d}h(t)$ admits a well defined logarithm $u(t) = \ln\bigl(t^{-d}h(t)\bigr)$, a holomorphic function in A unique modulo $2\pi i\mathbb{Z}$. Expanding u as in Lemma 17.14, we obtain two functions u_0, u_1 holomorphic in the respective disks $D_i \subset \mathbb{P}$. The exponents $g_i = \exp u_i$ of these functions are holomorphic, nonvanishing in U_i and satisfy the identity $t^{-d}h(t) = g_0/g_1$ on U_{01}. Rewriting this identity in the form

$$h(t) \cdot g_1(t) = g_0(t)\, t^d,$$

we prove that the holomorphic cocycles $\mathcal{L}$ and $\mathcal{L}_d$ are equivalent; cf. with (17.5). The equality $t^{d'}g_1 = g_0\, t^d$ with $d \neq d'$ is impossible, since the variation of argument of each holomorphic nonvanishing function g_i along the circle is zero, while that of the ratio $t^{d-d'}$ is $2\pi(d - d')$. □

Proposition 17.13 gives classification of scalar cocycles inscribed in the standard covering of the Riemann sphere $\mathbb{P}$ by two charts. In fact, this particular case suffices to describe *all* scalar cocycles, hence *all* holomorphic line bundles over $\mathbb{P}$.

Theorem 17.16. *Any line bundle over the Riemann sphere is holomorphically equivalent to the standard bundle ξ_d of some degree $d \in \mathbb{Z}$.*

Proof. We first show that any line bundle π_0 over the unit disk $\mathbb{D} \subset \mathbb{C}$ is equivalent to the trivial bundle. Indeed, consider the cocycle $\mathcal{L}$ which defines the bundle π_0. This cocycle is inscribed in a finite covering $\mathfrak{U}$. By further refinement of this covering we may assume that it is a "triangulation", i.e., the domains U_α are small ε-neighborhoods of triangles of some triangulation of $\mathbb{D}$ (one can also choose partition of the disk into small squares arranged in a grid). For our purposes it is important that the domains $U_1, \dots, U_N$ can be ordered in such a way that the intersections

$$U_{k+1} \cap (U_1 \cup \cdots \cup U_k), \qquad k = 1, 3, \dots, N-1,$$

are all connected and *simply connected*; see Fig. III.1.

Assume by induction that the cocycle $\mathcal{L}$ is solvable over $U' = U_1 \cup \cdots \cup U_k$. Then, replacing the cocycle $\mathcal{H}$ by an equivalent cocycle, we may assume that all transitions h_{ij} between domains with numbers $\leqslant k$ are trivial. We claim that the cocycle can be trivialized also over $U' \cup U$, $U = U_{k+1}$. Indeed, in this case all we have to show is that any holomorphic invertible function h in the intersection $V = U \cap U'$ can be represented as the quotient of two functions, $h = g/g'$, holomorphic and invertible in U and U' respectively.

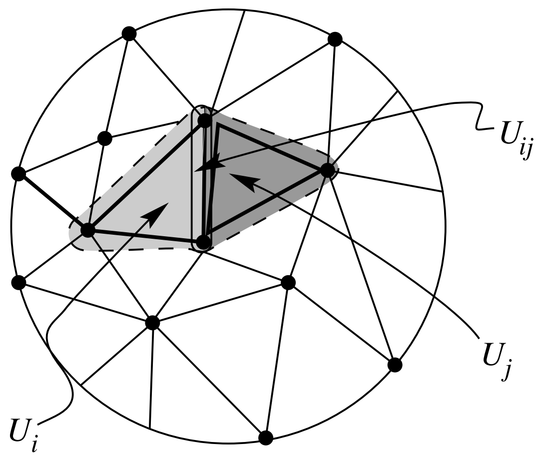

Figure III.1. Triangulation and "triangulated covering" of a disk

Since V by construction is simply connected, $\ln h$ is a well-defined holomorphic function which can be represented as a difference of two holomorphic functions by Lemma 17.14. After exponentiation we obtain the representation $h = g/g'$ and hence prove the solvability of the cocycle $\mathfrak{L}$ restricted on the union $U' \cup U = U_1 \cup \cdots \cup U_k \cup U_{k+1}$. By induction, the cocycle is solvable (and hence the corresponding line bundle π_0 is solvable).

Thus any holomorphic bundle over $\mathbb{P}$ can be trivialized over each of two charts of a standard Birkhoff–Grothendieck covering. This means that the problem of classification of *arbitrary* cocycles over $\mathbb{P}$ is reduced to classification of (scalar) Birkhoff–Grothendieck cocycles inscribed in a standard covering. By Proposition 17.13, each such cocycle is equivalent to a standard cocycle. □

17E. Sections of holomorphic vector bundles. Since the total space of a vector bundle is in general not a Cartesian product, we need a suitable generalization of the notion of vector functions.

Definition 17.17. A *section* of a holomorphic vector bundle $\pi\colon S \to T$ is a map $s\colon T \to S$, such that $\pi \circ s = \mathrm{id}$, i.e., such that the image of every point $a \in T$ belongs to the fiber $\pi^{-1}(a)$. We will specifically deal with continuous, *holomorphic* and *meromorphic* sections (the latter will be defined separately later).

Remark 17.18. Sometimes we will deal with "sections" defined only over some (open) subset U of the base T. In this case, to avoid confusion, we will say about *local sections*, explicitly specifying their domains.

Trivializations over domains U_α allow us to associate with every section s a holomorphic *vector cochain* $\{x_\alpha\}$, the collection of vector functions

$$x_\alpha : U_\alpha \to \mathbb{C}^n, \qquad x_\alpha = \Phi_\alpha \circ s\big|_{U_\alpha}.$$

Using a different trivialization Φ_β on the intersection of two domains of trivialization, replaces the function x_α by the function x_β,

$$x_\beta = H_{\beta\alpha} x_\alpha \qquad \text{on} \quad U_{\alpha\beta}. \tag{17.10}$$

Conversely, given a matrix cocycle $\mathcal{H} = \{H_{\alpha\beta}\}$ and a vector bundle defined by this cocycle, any holomorphic vector cochain $\{x_\alpha\}$ which satisfies the conditions (17.10) on the pairwise intersections, defines a section of the bundle.

However, not all bundles admit nontrivial (not identically zero) holomorphic sections (Problem 17.7).

Example 17.19. Sections of the tangent bundle $\mathbf{T}M$ are called (holomorphic) vector fields on M. Sections of the cotangent bundle are called (holomorphic) 1-forms. There are no globally defined holomorphic 1-forms without poles on the Riemann sphere $\mathbb{P}$ (otherwise their primitives would be globally defined holomorphic nonconstant functions), hence $\mathbf{T}^*\mathbb{P}$ does not admit holomorphic sections. Globally defined holomorphic vector fields on $\mathbb{P}$ do exist, but they must necessarily have zeros.

Absence of holomorphic sections motivates introduction of a slightly more general notion of a *meromorphic section* of a holomorphic bundle.

Definition 17.20. A *meromorphic* section of a holomorphic vector bundle defined by a holomorphic matrix cocycle $\mathcal{H} = \{H_{\alpha\beta}\}$, is a meromorphic vector cochain $\{x_\alpha\}$, $x_\alpha \in \mathcal{M}(U_\alpha) \otimes_{\mathbb{C}} \mathbb{C}^n$ which satisfies the identities (17.10) on the intersections $U_{\alpha\beta}$.

All meromorphic sections of a given bundle form an infinite-dimensional linear space over $\mathbb{C}$ and, moreover, a linear space over the field $\mathcal{M}(T)$ of meromorphic functions on the base T, since two sections can be added, and any meromorphic section can be multiplied by a meromorphic (scalar) function. The corresponding meromorphic vector cochains obey the obvious rules,

$$s = s' + s'' \iff x_\alpha = x'_\alpha + x''_\alpha, \qquad s' = \varphi \cdot s \iff x'_\alpha = \varphi x_\alpha.$$

The set of all meromorphic sections of a bundle $\pi : T \to S$ will be denoted by $\Gamma(\pi)$.

17F. Degree of a holomorphic bundle. Recall that the order of a meromorphic *scalar* function $\varphi \in \mathcal{M}(\mathbb{C}, 0)$ of a scalar argument $t \in (\mathbb{C}, 0)$ is the order (positive or negative) of its principal Laurent term, $\operatorname{ord}_0 \varphi = \nu$ if and only if $\varphi(t) = c_\nu t^\nu + c_{\nu+1} t^{\nu+1} + \cdots$, with $c_0 \neq 0$.

Definition 17.21. The order of a meromorphic *vector*-function $x(t) = \big(x_1(t), \dots, x_n(t)\big) \in \mathcal{M} \otimes \mathbb{C}^n$ is the minimal order of its components,

$$\operatorname{ord}_0 x = \min_{1 \leqslant j \leqslant n} \operatorname{ord}_0 .$$

One can instantly verify that $\operatorname{ord}_0 x(\cdot)$ is the unique integer number $d \in \mathbb{Z}$ such that $t^{-d} x(t)$ is holomorphic and nonvanishing at $t = 0$,

$$\operatorname{ord}_0 x(t) = d \iff t^{-d} x(t) \in \mathcal{O}(\mathbb{C}, 0) \otimes \mathbb{C}^n \text{ and } \lim_{t \to 0} t^{-d} x(t) \neq 0.$$

If π is a holomorphic vector bundle over a *one-dimensional* base (Riemann surface) T, then the *order* $\operatorname{ord}_a s$ of a meromorphic section $s \in \varGamma(\pi)$ at a given point $a \in T$ of the base can be defined as the order of the corresponding vector function x_α in any trivialization $U_\alpha \ni a$: since the transition cocycle consists of holomorphic matrix functions, this definition is self-consistent. The order is nonzero at all points except for a discrete set.

Proposition 17.22. *All nontrivial meromorphic sections of a line bundle over a compact Riemann surface T have the same total order*: *for any meromorphic section the sum*

$$\deg s = \sum_{a \in T} \operatorname{ord}_a s, \qquad s \in \varGamma(\pi), \tag{17.11}$$

is the same and depends only on the bundle π itself.

Proof. If the fibers are one-dimensional, then any two sections $s, s' \in \varGamma(\pi)$ are proportional, i.e., there exists a meromorphic *function* $\varphi \in \mathcal{M}(T)$ such that $s' = \varphi s$. Obviously, $\deg s' = \deg s + \sum_a \operatorname{ord}_a \varphi$, where the last term is the sum of orders of all poles and zeros of φ. Yet any meromorphic function φ considered as a map $\varphi \colon T \to \mathbb{P}$, assumes each value the same number of times (equal to the degree of this map). Applying this to the values 0 and ∞, we conclude that $\sum_a \operatorname{ord}_a \varphi = 0$, hence $\deg s = \deg s'$. □

Definition 17.23. The common degree of all meromorphic sections is called the *degree* of a line bundle π and denoted by $\deg \pi$.

The degree of arbitrary holomorphic vector bundle is *defined* as the degree of its determinant, the line bundle associated with the determinant cocycle (17.8),

$$\deg \pi = \deg(\det \pi). \tag{17.12}$$

We will need the following property of the degree. A holomorphic bundle map between bundles of the same dimension will be called *nondegenerate*, if it has a full rank at some point.

Lemma 17.24. *Let* $\pi\colon S\to T$ *and* $\pi'\colon S'\to T$ *be two bundles of the same rank over the same compact one-dimensional base* T.

If there exists a nondegenerate holomorphic bundle map $F\colon S\to S'$ *fibered over the identity map of the base, then* $\deg\pi\leqslant\deg\pi'$.

Proof. Consider first the case where S and S' are both *line* bundles defined by *scalar* cocycles $\mathcal{H}=\{h_{\alpha\beta}\},\mathcal{H}'=\{h'_{\alpha\beta}\}$ on trivializations over the same covering $\mathfrak{U}$, then a bundle map between them is defined by a collection of holomorphic functions $f_\alpha\not\equiv 0$ related to the cocycles $\mathcal{H},\mathcal{H}'$ by (17.7).

An arbitrary meromorphic section $s\in\varGamma(\pi)$ and its image $s'=Fs\in\varGamma(\pi')$ are defined by the meromorphic *scalar* cochains x_α and x'_α which satisfy the identity

$$x'_\alpha=f_\alpha x_\alpha. \tag{17.13}$$

Since $f_\alpha\not\equiv 0$, this implies that $\operatorname{ord}_a s'_\alpha=\operatorname{ord}_a s_\alpha+\operatorname{ord}_a f_\alpha\geqslant\operatorname{ord}_a s_\alpha$. Adding these inequalities over all points of T, we arrive at the inequality $\deg s'\geqslant\deg s$. By Proposition 17.22, this means that $\deg\pi'\leqslant\deg\pi$.

A general nondegenerate linear map $F\colon S\to S'$, represented by a matrix cochain $\{F_\alpha\}$, defines a nondegenerate map $\det F$ between the determinant bundles $\det\pi$ and $\det\pi'$. The map $\det F$ is defined by the nonzero scalar holomorphic cochain $f_\alpha=\det F_\alpha$: this follows (17.7) after passing to determinants and the definition of the determinant bundle (17.8). The lemma follows from the assertion for line bundles and the definition of degree of an arbitrary bundle. □

As a corollary, we may conclude that subbundles of a trivial bundle all have nonpositive degree.

Corollary 17.25. *Every subbundle of the trivial bundle over a compact Riemann curve has nonpositive degree.*

Proof of the corollary. Let $\pi\colon S\to T$ be a subbundle of rank n of the trivial bundle $\pi_0\colon T\times\mathbb{C}^{n+m}\to T$. We will prove that $\deg\pi\leqslant 0$. Indeed, one can always find a splitting of the fiber $\mathbb{C}^{n+m}=\mathbb{C}^n\oplus\mathbb{C}^m$ into two subspaces such that the fiber $\pi^{-1}(a)$ is transversal to $\mathbb{C}^m$ at some point $a\in T$. The projection on $\mathbb{C}^n$ parallel to $\mathbb{C}^m$ after restriction on the subbundle S becomes a holomorphic nondegenerate bundle map between π and the trivial subbundle $\pi'=T\times\mathbb{C}^n\to T$. By Lemma 17.24, $\deg\pi\geqslant\deg\pi'=0$. □

17G. Holomorphic and meromorphic connexions. If $x\colon T \to \mathbb{C}^n$ is a holomorphic vector function of one or several variables, then its differential is a vector-valued 1-form on T. Once fibers over different points of the base T are different, as in the case of holomorphic vector bundles, the notion of *derivation of a section* needs to be appropriately modified. The result is the notion of a *connexion*, or in full *meromorphic connexion on a holomorphic vector bundle*.

Connexions can be described axiomatically by their geometric properties. Denote by $\Lambda^1(T) \otimes_{\mathcal{M}(T)} \Gamma(\pi)$ the $\mathcal{M}(T)$-module of meromorphic fiber-valued 1-forms on the base T of a holomorphic vector bundle π, the tensor product is taken over the field of meromorphic functions $\mathcal{M}(T)$. By definition, a fiber-valued 1-form $\omega \otimes s$ can be evaluated on any meromorphic vector field $Z \in \mathcal{D}(T)$, and the result will be the meromorphic section $\varphi \cdot s \in \Gamma(\pi)$, $\varphi = \omega(Z)$. This object generalizes the notion of a vector-valued 1-form. Now we give a generalization of the exterior derivative for vector-valued functions. This is a differential operator called a *connexion* on the bundle.

Definition 17.26. A *meromorphic connexion* on a holomorphic vector bundle π is a $\mathbb{C}$-linear operator

$$\nabla\colon \Gamma(\pi) \to \Lambda^1(T) \otimes \Gamma(\pi)$$

which satisfies the Leibnitz rule:

$$\begin{aligned} \nabla(\lambda s + \lambda' s') &= \lambda \nabla s + \lambda' \nabla s', \qquad \forall s, s' \in \Gamma(\pi),\ \lambda, \lambda' \in \mathbb{C}, \\ \nabla(\varphi \cdot s) &= \varphi \cdot \nabla s + df \otimes s, \qquad \forall s \in \Gamma(\pi),\ \varphi \in \mathcal{M}(T). \end{aligned} \tag{17.14}$$

The result ∇s of a derivation is a fiber-valued 1-form on T.

Example 17.27. If π is a trivial bundle with $S = T \times \mathbb{C}^n$, then the standard (vector) exterior derivative

$$\nabla x = dx, \qquad \forall x\colon T \to \mathbb{C}^n$$

obviously satisfies the rules (17.14). In fact, for trivial bundles we can easily describe *all* differential operators satisfying the axioms (17.14). Indeed, if ∇, ∇' are two such operators, then their difference is a *linear operator on each fiber*: from (17.14) it immediately follows that

$$(\nabla - \nabla')(\varphi \cdot x) = \varphi \cdot [(\nabla - \nabla')x].$$

This means that the difference between the operators is defined by an $n \times n$-matrix-valued form: evaluated on a tangent vector at a point $a \in T$ of the base, it becomes a linear automorphism of the respective fiber $\pi^{-1}(a) \cong \mathbb{C}^n$ into itself.

In other words, any connexion ∇ *on the trivial bundle* can be represented using a suitable meromorphic matrix 1-form $\Omega \in \operatorname{Mat}(n, \Lambda^1(T) \otimes \mathcal{M}(T))$ as the difference $\nabla = d - \Omega$, that is,

$$\nabla x = dx - \Omega x, \qquad \forall x\colon T \to \mathbb{C}^n, \tag{17.15}$$

The matrix 1-form Ω is called the *connexion form* of the connexion ∇.

For arbitrary (nontrivial) bundles such characterization is true only locally, in trivializing charts.

17H. Connexions vs. linear systems. If $F\colon S \to S'$ is an invertible holomorphic bundle map between two bundles π, π' over the same base, then this map allows us to carry any connexion on S to a connexion on S' and vice versa. Two connexions ∇, ∇' on the two bundles are called F-*related*, if $F(\nabla s) = \nabla'(Fs)$ for any section $s \in \Gamma(\pi)$. Here by Fs is denoted the section $s' \in \Gamma(\pi')$ obtained by application of F to the section s.

Assume that both S, S' are trivial bundles (of the same rank) and F is a gauge map defined by the matrix function $F(a) \in \mathrm{GL}(n, \mathbb{C})$ as in (15.9). It transforms a vector function $a \mapsto x(a)$ into the vector function $x'(a) = F(a)x(a)$. Thus two connexions, $\nabla = d - \Omega$ and $\nabla' = d - \Omega'$, defined by two matrix forms Ω, Ω', are F-related if and only if $F(dx - \Omega x) = d(Fx) - \Omega' Fx$ for any vector-valued holomorphic function $x(\cdot)$. This condition is equivalent to the matrix identity

$$\Omega' = dF \cdot F^{-1} + F\Omega F^{-1} \tag{17.16}$$

which naturally coincides with the law of gauge transformation (15.10).

This observation allows us to represent any connexion on a holomorphic bundle by a collection of matrix 1-forms associated with different local trivializations of this bundle. Indeed, if Φ_α is a local trivialization of a holomorphic vector bundle with a meromorphic connexion ∇, then there exists a unique meromorphic connexion ∇_α on the trivial bundle $U_\alpha \times \mathbb{C}^n$, which is Φ_α-related to ∇. On the intersection of two charts $U_{\alpha\beta}$ two different trivializations lead to two different connexion 1-forms $\Omega_\alpha, \Omega_\beta$. By (17.16), these two matrix forms are related by the identity

$$dH_{\alpha\beta} = \Omega_\alpha H_{\alpha\beta} - H_{\alpha\beta}\Omega_\beta. \tag{17.17}$$

Conversely, given a collection of trivializations of a holomorphic vector bundle, related by a matrix cocycle $\mathcal{H} = \{H_{\alpha\beta}\}$ and an arbitrary collection of meromorphic matrix 1-form Ω_α satisfying the transition identities (17.16) on the pairwise intersections, we can define a meromorphic matrix connexion ∇ as the operator sending the vector cochain $\{x_\alpha\}$ defining an arbitrary section $s \in \Gamma(\pi)$ into the cochain $\{\theta_\alpha\}$ of vector-valued 1-forms $\theta_\alpha = dx_\alpha - \Omega x_\alpha$. It is a standard exercise to verify that if the initial cochain satisfies (17.10), then the cochain $\{\omega_\alpha\}$ defines a section of $\Lambda^1(T) \otimes \Gamma(\pi)$, i.e., satisfies the analogous identity $H_{\alpha\beta}\theta_\beta = \theta_\alpha$ on the pairwise intersections.

Describing a meromorphic connexion by its connexion (matrix) forms in suitable trivializations allows us to translate all notions and properties of theory of linear systems, which are invariant by holomorphic gauge equivalence, from the local theory of linear systems to the global context. We skip trivial checks.

Definition 17.28. The *singular locus* of a meromorphic connexion ∇ is defined as the collection of points at which the connexion matrix Ω_α in some (hence in any) trivialization has a pole. A meromorphic connexion is *holomorphic* if it has no poles.

Definition 17.29. A singular point of ∇ is *Fuchsian*, if it has a first order pole in some (hence in any) trivialization.

A singular point is *regular*, if it is regular for some (hence for any) linear system $dx = \Omega_\alpha x$.

For a Fuchsian connexion one can define its *residue* $\operatorname{res}_a \nabla$ at each Fuchsian singularity. This is a linear operator of the fiber $\pi^{-1}(a)$ into itself, defined in the local trivializing chart as the residue of the corresponding matrix connexion form:

$$\begin{gathered}\operatorname{res}_a \nabla \colon \pi^{-1}(a) \to \pi^{-1}(a), \\ \operatorname{res}_0(d - \Omega) = A \iff \Omega = (t^{-1}A + A_0 + A_1 t + \cdots)\, dt.\end{gathered} \tag{17.18}$$

A vector function $x(\cdot)$ whose differential vanishes, $dx(\cdot) \equiv 0$, is (locally) constant and its graph is a horizontal hyperplane in the cylinder $T \times \mathbb{C}^n$. Such horizontal hyperplanes allow us to identify between themselves any two fibers $\{t = a\}$ and $\{t = b\}$, if the corresponding points belong to the same horizontal hyperplane.

The analogous notions for general bundles are defined using the covariant derivative ∇ instead of the exterior derivative d.

Definition 17.30. A *horizontal section* for a connexion ∇ on a holomorphic vector bundle π is a section satisfying the differential equation $\nabla s = 0$.

If ∇ is a connexion on the trivial bundle $U \times \mathbb{C}^n$ with a connexion form Ω, then horizontal sections $t \mapsto x(t)$ satisfy the Pfaffian linear equation $dx - \Omega x = 0$. Thus we see that connexions correspond to globally defined linear systems introduced in a geometric (coordinate-free) way.

Remark 17.31. Existence of horizontal *local* holomorphic sections over any simply connected chart free from singular points of a connexion, is automatic *only in the case where the base T is complex one-dimensional.* In all other cases even local existence of horizontal sections is guaranteed only under certain condition of *flatness* (absence of the *curvature*) of the connexion; see Problem 17.13.

Linear systems	**Meromorphic connexions**
Domain T (Riemann surface)	Base of the bundle T
Vector functions $\mathfrak{M}(T) \otimes \mathbb{C}^n$	Sections of the bundle $\varGamma(\pi)$
Matrix 1-form $\Omega \in \operatorname{Mat}(n, \varLambda^1(T) \otimes \mathfrak{M}(T))$	Meromorphic connexion $\nabla \colon \varGamma(\pi) \to \varGamma(\pi) \otimes \varLambda^1(T)$
Solutions of the linear system $dx = \Omega x$	Horizontal sections $\nabla s = 0$
Holonomy (monodromy), Cauchy operators	Parallel transport between fibers
Gauge transform	Bundle map

Table III.1. Glossary of terms: meromorphic connexions on holomorphic vector bundles vs. linear systems

In the same way as solutions of linear systems, horizontal sections are usually *multivalued*, i.e., exist only on the universal cover of $T \smallsetminus \varSigma$, where $\varSigma = \operatorname{Sing} \nabla$ is the singular locus of the connexion. On the other hand, if the base T is one-dimensional, Theorem 15.3 implies that horizontal sections can be constructed over any simply connected domain in the punctured base $T \smallsetminus \varSigma$. Moreover, partition of S on horizontal sections defines a *horizontal foliation* $\mathcal{F}_\nabla$ (with singularities) of the total space S, transversal to all fibers over nonsingular locus $T \smallsetminus \varSigma$.

Horizontal sections are "locally constant" with respect to the connexion ∇ and hence can be used to define the *parallel transport* between nearby fibers $\pi^{-1}(a)$ and $\pi^{-1}(a')$ over two sufficiently close points $a, a' \in T$. This transport is the precise equivalent of the holonomy map between two cross-sections τ_a and $\tau_{a'}$ to the null leaf of the foliation defined by an arbitrary linear system (15.3). In the same way as for the linear systems (connexions on the trivial bundles), the parallel transport along the leaves of horizontal foliation defines the *holonomy group* of the connexion. All these notions for connexions on trivial bundles coincide with their previously introduced homologues for linear systems. Table III.1 provides a glossary of parallel terms.

Theorem 17.32. *Let $\pi \colon S \to T$ be a holomorphic vector bundle of rank n and ∇ a meromorphic connexion on this bundle with the singular locus $\varSigma$.*

Then for any point a, any linearly independent vectors in the fiber $\pi^{-1}(a)$ and any simply connected domain $U \subseteq T \smallsetminus \Sigma$ there exist n holomorphic sections of π over U, linearly independent in each fiber.

The parallel transport along horizontal sections over closed paths γ from the fundamental group $\pi_1(S \smallsetminus \Sigma, a)$ defines a representation $\gamma \mapsto \Delta_\gamma$ of this group by linear holonomy operators $\Delta_\gamma \in \mathrm{GL}(\pi^{-1}(a))$.

If π, π' are two bundles over the same base and F is a holomorphic or meromorphic bundle map between them fibered over the identity, and ∇, ∇' are two F-related connexions on these two bundles, then the corresponding holonomy groups are also F-related, i.e., conjugated[3] *by the linear map $F(a)\colon \pi^{-1}(a) \to \pi'^{-1}(a)$.* □

17I. Connexions on line bundles. Trace of a meromorphic connexion. Connexions on line bundles (of rank 1) are determined by the scalar meromorphic 1-forms ω_j in each trivialization, i.e., each connexion ∇ is determined by its cochain of *scalar* 1-forms $\{\omega_\alpha\}$. Since 1×1-matrices commute, on the overlapping of domains U_i and U_j of two different trivializations, two forms ω_i, ω_j differ by an additive holomorphic term, the logarithmic derivative of the transition cocycle,

$$\omega_i = d \ln h_{ij} + \omega_j, \qquad d \ln h_{ij} = dh_{ij}/h_{ij}. \tag{17.19}$$

In particular, the residue $\operatorname{res}_a \nabla$ is well defined as the *scalar* residue of any of the two forms,

$$\operatorname{res}_a \nabla = \operatorname{res}_a \omega_i = \operatorname{res}_a \omega_j, \qquad a \in U_{ij}.$$

The total of residues of any meromorphic 1-form on a compact Riemann surface is zero: the sum makes sense since the residues are (complex) numbers that can be added between themselves. The following is a generalization of this fact for arbitrary line bundles.

Theorem 17.33. *The total of residues of any meromorphic connexion on a line bundle π over a compact Riemann surface T is the same for all connexions and equal to the degree of the bundle,*

$$\sum_{a \in T} \operatorname{res}_a \nabla = \deg \pi.$$

Proof. The difference between any two meromorphic connexions ∇, ∇' on the same line bundle is a globally well-defined meromorphic 1-form $\eta = \nabla - \nabla' \in \Lambda^1(T)$. Indeed, by (17.15) the difference is a well-defined operator-valued 1-form, but every linear self-map from $\mathrm{GL}(1, \mathbb{C})$ can be identified with

[3] In particular, if a point $a_j \in T$ is singular for one connexion and nonsingular for another, then the holonomy operators corresponding to a simple loop around this point are both trivial (identical).

its multiplicator which is a complex number (rather than an element of some fiber). From this observation it obviously follows that

$$\sum_a \operatorname{res}_a \nabla - \sum_a \operatorname{res}_a \nabla' = \sum_a \operatorname{res}_a \eta = 0,$$

since the total of residues of any meromorphic 1-form is zero (the total of integrals of η along all small loops around all singularities on T). Thus the sum of residues indeed does not depend on the connexion.

To show that it is equal to the degree, consider an arbitrary meromorphic section $s \in \varGamma(\pi)$ defined by a holomorphic cochain, $s \sim \{x_\alpha\}$, and let ∇ be the unique connexion for which s is horizontal (see Exercise 17.6). This connexion is defined by the cochain of logarithmic derivatives $\{\omega_\alpha\}$,

$$\nabla \cong \{\omega_\alpha\}, \qquad \text{where} \quad \omega_\alpha = dx_\alpha \cdot x_\alpha^{-1}.$$

The residue of the connexion ∇ at any point is the order of the section s at this point. Therefore

$$\sum_a \operatorname{res}_a \nabla = \sum_a \operatorname{res}_a \omega_\alpha = \sum_a \operatorname{ord}_a x_\alpha = \sum_a \operatorname{ord}_a s = \deg \pi$$

by (17.11). □

This result cannot be directly generalized to connexions on arbitrary bundles of rank greater than 1, since for such bundles the residues are linear self-maps of different fibers, hence cannot be simply added together. Thus the "total of all residues" for an arbitrary connexion is meaningless. The best one can get is a formula for the "total of *traces* of all residues" which is defined as follows.

Any meromorphic connexion ∇ on a holomorphic vector bundle π induces the *trace connexion*, denoted by $\operatorname{tr}\nabla$, on the determinant bundle $\det\pi$. If the connexion ∇ is trivialized by a cochain of meromorphic matrix 1-forms $\{\Omega_\alpha\}$, then $\operatorname{tr}\nabla$ is trivialized by the cochain $\{\omega_\alpha\}$,

$$\nabla \cong \{\Omega_\alpha\} \overset{\text{def}}{\iff} \operatorname{tr}\nabla \cong \{\operatorname{tr}\Omega_\alpha\}. \tag{17.20}$$

Proposition 17.34. *The connexion* $\operatorname{tr}\nabla$ *is a well-defined meromorphic connexion on the bundle* $\det\pi$*.*

Two connexions ∇ *and* $\operatorname{tr}\nabla$ *are* det*-related: if* $s_1, \dots, s_n$ *are* n *linearly independent meromorphic sections of a rank* n *bundle* π*, horizontal for* ∇*, then their wedge product* $s_1 \wedge \cdots \wedge s_n$ *is a section of the line bundle* $\det\pi$*, horizontal for the connexion* $\operatorname{tr}\nabla$*.*

Both connexions have the same singular locus, and at every singular point

$$\operatorname{res}_a \operatorname{tr}\nabla = \operatorname{tr}\operatorname{res}_a \nabla. \tag{17.21}$$

Proof. To prove the first assertion, consider the cocycle $\mathcal{H} = \{H_{\alpha\beta}\}$ defining π and the respective cocycle $\det \mathcal{H} = \{h_{\alpha\beta}\}$, $h_{\alpha\beta} = \det H_{\alpha\beta}$. By the Liouville–Ostrogradskii formula (Problem 15.10),

$$\operatorname{tr}\Omega_\beta = \operatorname{tr}(dH_{\beta\alpha} \cdot H_{\alpha\beta}) + \operatorname{tr}(H_{\beta\alpha}\Omega_\alpha H_{\alpha\beta}) = dh_{\beta\alpha} \cdot h_{\alpha\beta} + \operatorname{tr}\Omega_\alpha,$$

that is, the cochain $\{\operatorname{tr}\Omega_\alpha\}$ representing $\operatorname{tr}\nabla$, is indeed a connexion on the bundle defined by the cocycle $\det \mathcal{H}$.

If $\{X_\alpha(t)\}$ is a fundamental (multivalued) matrix solution associated with the sections $s_1, \dots, s_n$, then $\{u_\alpha\} = \{\det_\alpha X(t)\}$ is a cochain defining the corresponding section of $\det \pi$. By the Liouville–Ostrogradskii formula,

$$\Omega_\alpha = \dot{X}_\alpha \cdot X_\alpha^{-1}, \qquad \operatorname{tr}\Omega_\alpha = \dot{u}_\alpha / u_\alpha,$$

which proves that the two connexions are det-related. □

By definition of degree of the arbitrary bundle, we have an immediate corollary from Theorem 17.33 and Proposition 17.34.

Corollary 17.35 (Index theorem for connexions on a vector bundle)**.** *For any meromorphic connexion π on a holomorphic vector bundle π over a compact Riemann surface,*

$$\sum_a \operatorname{res}_a \operatorname{tr}\nabla = \sum_a \operatorname{tr}\operatorname{res}_a \nabla = \deg \pi. \quad \square \tag{17.22}$$

17J. Classification of holomorphic vector bundles over $\mathbb{P}$. We conclude this section by a complete description of all holomorphic vector bundles over the Riemann sphere.

Theorem 17.36. *Any holomorphic vector bundle over the open unit disk $\mathbb{D}$ or the affine line $\mathbb{C}$, is holomorphically trivial.*

Theorem 17.37. *Any holomorphic vector bundle π over the Riemann sphere $\mathbb{P}$ is holomorphically equivalent to the direct sum of standard line bundles of different degrees*

$$\xi_D \overset{\text{def}}{=} \xi_{d_1} \oplus \cdots \oplus \xi_{d_n}, \qquad D = \operatorname{diag}\{d_1, \dots, d_n\},\ d_i \in \mathbb{Z}.$$

The collection of integer numbers $\{d_1, \dots, d_n\}$, called the splitting type*, is defined by the bundle uniquely modulo permutation.*

These results will be derived from assertions on solvability and equivalence of matrix cocycles.

Consider first the simplest cocycles inscribed in a covering by two charts $U_0, U_1 \subset \mathbb{P}$ (they may not cover the entire sphere $\mathbb{P}$). Assume that both U_i are topological disks with piecewise-smooth boundaries in $\mathbb{C}$ and their intersection U_{01} is connected.

There are then two topologically different possibilities: either the intersection U_{01} is also a topological disk (bounded by piecewise-smooth curve), or U_{01} is a topological annulus.

In the first case the holomorphic cocycle inscribed in such a covering will be referred to as a *Cartan cocycle*.

Lemma 17.38. *Any Cartan cocycle is holomorphically solvable.*

Matrix cocycles inscribed in the covering of the second type, in which the intersection U_{01} is a topological annulus, will be referred to as the Birkhoff–Grothendieck cocycle, cf. with §17**D**. Without loss of generality we will assume that the covering is standard (formed by two circular disks centered at $t = 0$ and $t = \infty$ respectively).

Lemma 17.39. *Any Birkhoff–Grothendieck matrix cocycle* $\mathcal{H} = \{H_{01}, H_{10}\}$ *is equivalent to a Birkhoff–Grothendieck cocycle defined by the diagonal matrix function* $\{t^D, t^{-D}\}$ *with an integer diagonal matrix* $D = \operatorname{diag}\{d_1, \dots, d_n\}$.

In other words, Lemma 17.39 asserts that any holomorphic function $H_{01}(t)$ in the annulus $U_{01} = A$ admits factorization

$$H_{01}(t) = F_0(t) \cdot \begin{pmatrix} t^{d_1} & & \\ & \ddots & \\ & & t^{d_n} \end{pmatrix} \cdot F_1(t) \tag{17.23}$$

with the matrix functions F_0, F_1 holomorphic and invertible in the disks U_0, U_1 around $t = 0$ and $t = \infty$ respectively.

This very deep result can be viewed from different angles. The treatment based on the operator theory and integral equations can be found in the article [**GK60**]. In this article the authors construct the factorization (17.23) of a matrix function H_{01} of very weak regularity (defined on the circle $|t| = 1$ and merely integrable on it), and obtain the factors $F_{0,1}$ holomorphic invertible inside and outside this circle, so that the identity (17.23) is understood on the circle in the sense of the limit values.

An alternative approach uses methods of analytic matrix functions. The first step is to show that any cocycle can be solved in meromorphic rather than in holomorphic functions. In other words, we show that there are no analytic (nonalgebraic) obstructions for solvability of matrix cocycles.

Theorem 17.40. *Any Cartan or Birkhoff–Grothendieck cocycle is meromorphically solvable*: *there exists a pair of meromorphic and meromorphically invertible matrix functions* F_i *defined in the domains* U_i, $i = 0, 1$, *such that*

$$F_0 = H_{01} F_1 \qquad \text{on the intersection} \quad U_{01}. \tag{17.24}$$

Idea of the proof. In the noncommutative (matrix) case one cannot reduce the "multiplicative" matrix equation (17.24) to the "additive" equation by simply taking logarithms as in the proof of Proposition 17.13. Yet if H_{01} is a near identical cocycle, $H_{01} = E + \varepsilon H$ for a small parameter ε, then one can use the ansatz $F_i = E + \varepsilon G_i$, $i = 0, 1$, and "linearize" the equation (17.24), rewritten as $\varepsilon G_0 = \varepsilon(H + G_1) + O(\varepsilon^2)$, by keeping only terms of first order in ε. This linearized equation $G_0 = H + G_1$ is additive and can always be solved with respect to G_0, G_1 in holomorphic matrix functions by literally reproducing the proof of Lemma 17.14. From this solvability after some additional efforts one can derive *holomorphic solvability of the matrix equation* (17.24) for all *near-identical* holomorphic matrix cocycles. This step resembles solving a nonlinear integral equation using the resolvent of a linearized equation. Somewhat unexpectedly, the resolvent operation for the Birkhoff–Grothendieck case is bounded and the corresponding nonlinear equation can be solved using contracting mapping principle. In the Cartan case the resolvent operator (given by the Cauchy integral) is unbounded and one has to use an appropriate modification of the Newton–Kolmogorov method of accelerated convergence to overcome this difficulty.

Once the problem is solved for any near-identical cocycles, any other (not necessarily near-identical) matrix cocycle can be approximated with any specified accuracy by a *rational matrix cocycle*. The rational cocycles are obviously meromorphically solvable (it is sufficient to collect factors with poles in the corresponding charts). From this observation one can easily derive meromorphic solvability of an arbitrary cocycle.

Accurate demonstrations can be found in [**GR65**, §VI.E], [**AB94**, §3.3] and in the recent book [**Bol00**, Lecture 9]. □

The second part of the proof transforms meromorphic solution of a cocycle into holomorphic solution of this cocycle or a into a holomorphic conjugacy of it with some standard cocycle. It is this step in which the difference between noncompact ($\mathbb{C}$ or $\mathbb{D}$) and compact ($\mathbb{P}$) base plays the key role. We will derive Lemmas 17.38 and 17.39 from Theorem 17.40 by elementary row and column operations with matrix functions.

Recall that an elementary operation on rows of a matrix is one of the following three:

(1) transposition of two rows of a matrix,

(2) adding to one of the rows a linear combination of other rows,

(3) multiplication of a row by a nonzero scalar.

Each elementary operation can be achieved by the left multiplication of the matrix by an appropriate *elementary matrix*. Except for the third type, the determinant of the corresponding elementary matrix is 1. Three parallel elementary operations on columns of a matrix can be achieved by an appropriate right multiplication.

In an obvious way, these elementary operations can be generalized for meromorphic matrix functions: transformations of the second type consist of adding to a row of a matrix function a linear combination of other rows with meromorphic coefficients. Transformations of the third type consist of multiplication of a row by a nonzero meromorphic function. Elementary operations on columns of meromorphic matrix functions are also self-explanatory.

Proof of Lemma 17.38. By Theorem 17.40, any Cartan cocycle can be resolved by a *meromorphic* cochain $\{F_0, F_1\}$. We will implement a series of modifications transforming this meromorphic cochain to a holomorphic cochain.

First, the meromorphic cochain can be modified so that all matrix functions $F_i(t)$ become holomorphic in the corresponding domains $U_i \subseteq \mathbb{C}$. To that end, all functions $F_i(t)$ should be multiplied by a suitable scalar power $(t - t_k)^{\nu_k}$, $\nu_k \in \mathbb{N}$, for each finite pole t_k of order ν_k. Clearly, the determinants of the holomorphic matrices $F_i(t)$ obtained by such multiplication, remain not identically vanishing, though they still may have isolated zeros of finite order.

In order to get rid of these zeros, we will further multiply F_i simultaneously by rational matrix functions from the right (this operation obviously will preserve the identity $H_{01}F_1 = F_0$). If t_* is an isolated root of, say, $\det F_1(t)$, then one of the columns of the matrix $F_1(t_*)$ is a linear combination of other columns, so that after the right multiplication by an appropriate constant matrix C one of the columns of $F_1(t_*)$ becomes zero. Then all entries from this column of the matrix function $F_1(t)C$ have the common factor $(t - t_*)$. After the right multiplication by the rational matrix function $R(t) = \operatorname{diag}\{1, \dots, (t - t_*)^{-1}, \dots, 1\}$, the modified matrix function $F_1(t)CR(t) = F_1'(t)$ remains holomorphic at t_*, and so apparently is $F_0'(t) = H_{01}(t)F_1'(t) = F_0(t)CR(t)$.

The total number of zeros of $\det F_i'(t)$, counted with multiplicities in $\mathbb{C}$, will decrease by 1 compared to that of $\det F_i(t)$. After a finite number of such steps we will get rid of all zeros of the determinant. The resulting cochain will resolve the cocycle, since by definition of the Cartan cocycle, both U_0 and U_1 belong to the finite part $\mathbb{C}$. The proof of Lemma 17.38 is complete. $\square$

The proof of Lemma 17.39 requires the following result, known as the Sauvage lemma [**Har82**]. Let $(\mathbb{P}, \infty)$ be a small circular neighborhood of infinity. Any matrix function $H(t) = H_{01} \in \operatorname{Mat}(n, \mathcal{M}(\mathbb{P}, \infty))$, meromorphic and not identically zero in this neighborhood, can be considered as a *cocycle* on the covering of the Riemann sphere by two charts, $U_0 = \mathbb{C}$ and $U_1 = (\mathbb{P}, \infty)$, which intersect by the punctured disk, itself a limit case of an annulus. We will refer to such cocycle as a *Sauvage cocycle*

Lemma 17.41 (Sauvage lemma)**.** *Any Sauvage cocycle is holomorphically equivalent to a standard matrix cocycle* $\{t^D\}$ *with an appropriate diagonal integer matrix* D*, inscribed in the same covering.*

Proof. The proof is achieved by a series of suitable monopole gauge transforms which realize elementary matrix transformations bringing the Sauvage cocycle to a diagonal form.

1. If the germ $H(t)$ is holomorphic at $(\mathbb{P}, \infty)$ and degenerate at $t = \infty$, then there exists a *constant upper-triangular* matrix C and a holomorphic germ $H'(t)$ such that

$$CH(t) = t^{D'} H'(t), \qquad D' = \operatorname{diag}\{0, \dots, -1, \dots, 0\}. \tag{17.25}$$

Indeed, if $\det H(\infty) = 0$, then the rows of the *constant* matrix $H(\infty)$ must be linearly dependent, in particular, some row of it must be equal to a linear combination of the subsequent (relatively lower) rows. In other words, there exists an *upper*-triangular constant matrix C with determinant 1, such that the matrix $CH(\infty)$ has a zero row. But then this same row of the matrix function $CH(t)$ is divisible by t^{-1}, so that the matrix $H'(t) = t^{-D'} CH(t)$ is holomorphic at $t = \infty$.

Clearly, the order of zero of $\det H'(t)$ is strictly inferior (by one less) than the order of zero of $\det H(t)$:

$$\operatorname{ord}_\infty \det H'(t) = \operatorname{ord}_\infty \det H(t) - 1. \tag{17.26}$$

2. If D is an integer diagonal matrix $D = \operatorname{diag}\{d_1, \dots, d_n\}$ with non-increasing entries $d_1 \geqslant \cdots \geqslant d_n$, and $H(t)$ is holomorphic and degenerate at infinity, then the product $t^D H(t)$ is monopole equivalent to $t^{D+D'} H'(t)$ with D' and $H'(t)$ as above.

Indeed, by Step 1, there exists a constant upper-triangular matrix C such that $CH(t) = t^{D'} H'(t)$ with holomorphic $H'(t)$ satisfying (17.26). Consider the conjugacy of C by t^D, $\Pi(t) = t^D C\, t^{-D}$. Because of the upper-triangularity of C and monotonicity of the sequence $\{d_i\}$, the matrix function $\Pi(t)$ is an upper-triangular monopole. Since D and D' commute,

$$\Pi(t)\, t^D H(t) = t^D C\, t^{-D} \cdot t^D H = t^D C H = t^D t^{D'} H' = t^{D+D'} H'.$$

3. For an arbitrary diagonal matrix D one can find a *constant* permutation matrix $P \in \mathrm{GL}(n, \mathbb{C})$ (particular case of monopole) such that the diagonal entries of $D' = P t^D P^{-1}$ will be monotonous as required in Step 2. This shows that the condition on the order of the diagonal entries d_i, imposed in Step 2, can be always achieved by a suitable monopole equivalence (left multiplication by P):

$$P\, t^D H = P\, t^D P^{-1} \cdot PH = t^{D'} H',$$

with a holomorphic H' degenerate at infinity together with H.

4. The proof of the Sauvage lemma can be achieved by simple induction. Any meromorphic germ $H(t)$ can be represented as $t^{D_1}H_1(t)$ with $H_1(t)$ holomorphic at infinity: it is sufficient to multiply $H(t)$ by a suitable (scalar) power of t. Since $\det H(t) \not\equiv 0$, the multiplicity of the root of $\det H_1(t)$ at $t = \infty$ is finite. The inductive application of the construction described above in Steps 1–3, allows us to construct a sequence of monopole transformations reducing $H_1(t)$ to the form of a product of two terms, $t^{D_k}H_k(t)$ as above (diagonal and holomorphic at infinity respectively), with strictly decreasing orders of the roots $\operatorname{ord}_\infty \det H_k(t)$. After finitely many steps the holomorphic term $H_m(t)$ becomes nondegenerate at infinity, and the Sauvage lemma is proved. □

Proof of Lemma 17.39. Proceeding as in the proof of Lemma 17.38, we may assume without loss of generality that the Birkhoff–Grothendieck cocycle H_{01} is already solved by the meromorphic cochain $\{F_0, F_1\}$ such that both F_0, F_1 are holomorphic and holomorphically invertible everywhere in their domains, possibly except for the point $t = \infty$, where F_1 has a finite order pole.

By the Sauvage lemma 17.41, the meromorphic matrix function germ $F_1^{-1}(t)$ can be represented as a composition $F_1^{-1} = \varPi(t)\, t^D G(t)$ with a polynomial and polynomially invertible (monopole) function $\varPi(t)$ and holomorphically invertible germ $G(t)$ at $t = \infty$. The matrix function $G_1 = t^{-D}\varPi^{-1}F_1^{-1}$ defined on the entire domain U_1, is holomorphic and holomorphically invertible in this domain. Indeed, since terms in the latter equality are holomorphically invertible in $U_1 \smallsetminus \{\infty\}$, while at the point $t = \infty$ the germ of this composition is G. Substituting the expression for $F_1 = G_1^{-1}t^{-D}\varPi^{-1}$ into the identity $H_{10}(t)F_0(t) = F_1(t)$, we get

$$H_{10}F_0 = G_1^{-1}t^{-D}\varPi, \qquad \text{i.e.,} \qquad H_{10}F_0\varPi^{-1} = G_1^{-1}\, t^{-D}.$$

In other words, the holomorphic cochain $\{F_0\varPi^{-1}, G_1^{-1}\}$, conjugates the initial Birkhoff–Grothendieck cocycle $\mathcal{H} = \{H_{01}\}$ with the standard cocycle $\{t^{-D}\}$. □

Proof of Theorems 17.36 and 17.37. The proof of both these theorems is achieved by literally the same arguments as the proof of Theorem 17.16. Namely, we consider a “triangulated” covering and consecutively resolve the Cartan cocycles using Lemma 17.38, until the disk $\mathbb{D}$ is exhausted. In the case of the Riemann sphere $\mathbb{P}$ we can replace the initial cocycle by an equivalent Birkhoff–Grothendieck cocycle. Then Lemma 17.39 proves that this cocycle is equivalent to one of the standard cocycles corresponding to the vector bundle ξ_D.

It remains only to prove the uniqueness of the splitting type D (clearly, bundles with permuted linear subbundles are equivalent). Assume that there exists a holomorphic bundle map between two bundles of different types D and D'. Then there exist a holomorphic matrix cochain $\{H_0, H_1\}$ inscribed in the Birkhoff–Grothendieck covering, such that

$$H_1 = t^D H_0 \, t^{-D'}, \qquad H_i \in \mathrm{GL}(n, \mathcal{O}(U_i)).$$

Consider an arbitrary matrix element of the form $h^0_{ij}(t)\, t^{d_i - d'_j}$ in the right hand side. If $d_i \geqslant d'_j$, then this element is holomorphic both in U_0, since h^0_{ij} is holomorphic there, and in U_1, since it is equal to $h^1_{ij}(t) \in \mathcal{O}(U_1)$. This is possible only if h^0_{ij} is a constant, necessarily zero if $d_i > d'_j$.

Assume that the two tuples of numbers $d_1, \dots, d_n$ and $d'_1, \dots, d'_n$ are both arranged in the nonincreasing order. Consider the largest elements d_1 and d'_1. If $d_1 > d'_1$, then the matrices H_0, H_1 will have identically zero first row, contrary to their nondegeneracy. For reasons of symmetry, the strict inequality $d'_1 < d_1$ is also impossible. This leaves only one possibility, $d_1 = d'_1$. Let k be the first place when the numbers d_k and d'_k are different.

If $d_k > d'_k$, then the matrix function $H_0(t)$ is block-upper-triangular with the upper $k \times k$-block having identically zero last row. Such a matrix is identically degenerate contrary to the assumption on the cochain $\{H_0, H_1\}$. Thus $d_k \leqslant d'_k$. For reasons of symmetry we also have $d'_k \leqslant d_k$, i.e., $d_k = d'_k$.

In other words, after arranging in the same nonincreasing order, both splitting types D and D' must coincide; but this means that they are permutations of each other. □

Exercises and Problems for §17.

Problem 17.1. Let $h_\alpha \colon U_\alpha \to \mathbb{C}^n$ be an atlas of charts for some open covering $\mathfrak{U}$ of a holomorphic manifold M. Write explicitly the trivializations for the tangent and cotangent bundles $\mathbf{T}M$ and $\mathbf{T}^*M$.

Exercise 17.2. Prove that two equivalent holomorphic cochains define two holomorphic equivalent vector bundles over the same base.

Problem 17.3. Let $\mathcal{H}, \mathcal{H}'$ be two cocycles (of different size matrices) corresponding to the vector bundles S, S' respectively. Construct explicitly the cocycles associated with the direct sum $S \oplus S'$ and the tensor product $S \otimes S'$.

Problem 17.4. Let $S' \subset S$ be a subbundle. Prove that the cocycle $\mathcal{H}$ associated with S, is equivalent to a cocycle of block upper-triangular matrices. Describe the cocycle associated with the quotient bundle $S'' = S/S'$.

Exercise 17.5. Prove that among all cocycles $\{t^d\}$ on the Riemann sphere, only the cocycle with $d = 0$ is solvable.

Exercise 17.6. Prove that for a given meromorphic section of a line bundle, there exists a unique connexion for which this section is horizontal.

Problem 17.7. Prove that the line bundle ξ_d over the projective line $\mathbb{P}$ admits nontrivial holomorphic sections if and only if its degree d is nonnegative.

Problem 17.8. Prove that the line bundle ξ_d over the projective line $\mathbb{P}$ admits nontrivial automorphisms different from multiplication by a constant factor, if and only if its degree d is negative.

Problem 17.9. Prove that the tangent bundle $\mathbf{T}\mathbb{P}$ and cotangent bundle $\mathbf{T}^*\mathbb{P}$ over the Riemann sphere have degrees 2 and -2 respectively.

Problem 17.10. Prove that a holomorphic bundle of rank n admits n holomorphic sections linearly independent in each fiber, if and only if the bundle is equivalent to the trivial one.

Exercise 17.11. Prove from the definition that the notion of connexion is local. More precisely, prove that for any two meromorphic sections s, s' of the same bundle, both holomorphic at a point $a \in T$ and with the same 1-jet, the respective values coincide, $\nabla s(a) = \nabla s'(a) \in \pi^{-1}(a)$.

Exercise 17.12. Prove that any connexion on a bundle of rank n is completely determined by n linearly independent horizontal sections: if two connexions have n common horizontal sections, then they coincide as differential operators.

Problem 17.13. Let $\pi_0 : T \times \mathbb{C}^n \to T$ be a trivial bundle over a simply connected holomorphic manifold T and ∇ a holomorphic connexion on it (holomorphic means meromorphic without singularities).

Prove that a collection of n *horizontal* holomorphic sections linearly independent in each fiber over a neighborhood U of a point a exists if and only if the connexion matrix form $\Omega = (\omega_{ij})_{i,j=1}^n$, $\omega_{ij} \in \Lambda^1(T) \otimes \mathcal{M}(T)$, satisfies the equation

$$d\Omega - \Omega \wedge \Omega = 0, \tag{17.27}$$

in a neighborhood of the point a, where

$$d\Omega = (d\omega_{ij})_{i,j=1}^n, \qquad \Omega \wedge \Omega = \Big(\sum\nolimits_k \omega_{ik} \wedge \omega_{kj}\Big)_{i,j=1}^n$$

are two matrix-valued 2-forms on T.

Problem 17.14 ([**Bol00**])**.** Find the splitting type (collection of the indices $d_1, \dots, d_n$) for the bundles defined by the Birkhoff–Grothendieck cocycles

$$\begin{pmatrix} t & \lambda \\ & t^{-1} \end{pmatrix}, \qquad \begin{pmatrix} t & \\ \lambda & t^{-1} \end{pmatrix} \tag{17.28}$$

Problem 17.15. Let $\mathcal{H}$ be a holomorphically solvable Birkhoff–Grothendieck cocycle (say, rational). Prove that any other rational cocycle $\mathcal{H}'$ sufficiently close to $\mathcal{H}$ in the annulus $A = U_0 \cap U_1$, is also solvable.

Problem 17.16 (Yu. L. Shmul'yan, 1954)**.** Assume that the splitting type $d_1 \leqslant \dots \leqslant d_n$ of a Birkhoff–Grothendieck cocycle $\mathcal{H}$ has at most one gap, i.e., $d_n - d_1 \leqslant 1$. Prove that any close cocycle has the same splitting type. Give an example showing that this is not necessarily the case if $d_n - d_1 > 1$.

Exercise 17.17. Prove that the degree of the bundle ξ_D is equal to $|D| = d_1 + \cdots + d_n$.

Problem 17.18. Prove that any holomorphically invertible matrix function $F(t)$ in the annulus $A = U_0 \cap U_1$ can be factored out as $F(t) = H_0(t)H_1(t)\,t^D$ with the terms $H_i(t)$ holomorphically invertible in U_i, $i = 0, 1$, and an integer diagonal matrix D. It is this form that is sometimes called the *Birkhoff factorization*.

In particular, any nonzero meromorphic germ of a matrix function $F(t)$ at the infinity admits factorization $F(t) = \Pi(t)H(t)\,t^D$ with a monopole $\Pi(t)$ and a holomorphically invertible germ $H(t)$ at infinity.

Problem 17.19 ([**Bol00**]). Prove that a holomorphic vector bundle $\pi\colon S \to T$ is *topologically* trivial if and only if its degree is equal to zero. The topological triviality means that there exists a *homeomorphism* $F\colon S \to T \times \mathbb{C}^n$ fibered over the identity and linear on each fiber.

18. Riemann–Hilbert problem

> The problem is as follows: *To show that there always exists a linear differential equation of the Fuchsian class, with given singular points and monodromy group.* The problem requires the production of n functions of the variable z, regular throughout the complex z-plane except at the given singular points; at these points the functions may become infinite of only finite order, and when z describes circuits about these points the functions shall undergo the prescribed linear substitutions.
>
> *D. Hilbert*, 1901, reprinted from [**Hil00**]

The *Riemann–Hilbert problem*, also known as Hilbert's twenty-first problem, requires constructing a linear system with the prescribed monodromy group and positions of all singularities. The original formulation by Hilbert is somewhat confusing, since the clarification given in the text after it, describes only the regularity condition, while the main formulation explicitly mentions Fuchsian systems.

One can think of not one but rather *three* different accurate formulations, when a given monodromy group is required to be realized by:

(i) a Fuchsian linear nth order differential equation,

(ii) a linear system having only regular singularities, or

(iii) a Fuchsian system on the whole Riemann sphere $\mathbb{P}$.

In each case it is required that the equation (resp., the system) be nonsingular outside the preassigned points.

The negative answer in the first problem was known already by A. Poincaré: the reason is that the dimension of the space of all Fuchsian equations having m prescribed singular points on $\mathbb{P}$, is strictly smaller than the dimension of all admissible monodromy data, except for the case of second order