

## 16. Local theory of regular singular points and applications

In this section we consider linear systems defined by the *germs* of meromorphic 1-forms  $\Omega = A(t) dt$  at the origin which is a singular point  $t \in (\mathbb{C}, 0)$  of finite order  $r + 1$ . Such a germ will be referred to as a *singular point* of a linear system or simply a *singularity*.

The fundamental group of the punctured neighborhood  $(\mathbb{C}, 0) \setminus \{0\}$  is infinite cyclic, generated by a single loop  $\gamma_0$  going counterclockwise around the origin. The corresponding operator of analytic continuation will be denoted by  $\Delta$ . In a similar way indication of the loop will be omitted in the notations for the monodromy matrix

$$\Delta X(t) = X(t)M, \quad M \in \mathrm{GL}(n, \mathbb{C}). \quad (16.1)$$

The notion of gauge equivalence (holomorphic or meromorphic) can be easily localized so that one can speak about (locally) *holomorphically (meromorphically) equivalent singularities* of linear systems. More precisely, this means that we consider gauge maps of the “infinitely narrow cylinder”  $(\mathbb{C}, 0) \times \mathbb{C}^n$  into itself, having the form (15.9), where  $h: (\mathbb{C}, 0) \rightarrow (\mathbb{C}, 0)$  is the identical germ and the invertible matrix function  $H(t)$  belongs to  $\mathrm{GL}(n, \mathcal{O}(\mathbb{C}, 0))$  or  $\mathrm{GL}(n, \mathcal{M}(\mathbb{C}, 0))$  respectively. Furthermore, we can consider *formal* gauge transforms, when  $H$  is a *formal matrix series*,  $H \in \mathrm{GL}(n, \mathbb{C}[[t]])$ . Such formal gauge transforms naturally act on *formal linear systems* defined by formal Pfaffian equations

$$t^{r+1} dx = \Omega x, \quad \Omega = A(t) dt, \quad A \in \mathrm{Mat}(n, \mathbb{C}[[t]]).$$

Our immediate goal in this section is to give a local classification (holomorphic, meromorphic or formal) of singularities of linear systems. It turns out that for a special class of singularities, so-called *regular* singularities, the problem admits complete solution.

**16A. Regular singularities.** A pole of an analytic function  $f(t)$  can be described as an isolated singular point at which the absolute value  $|f(t)|$  grows at most polynomially in  $|t|^{-1}$  (assuming the singular point at the origin). This moderate growth condition ensures numerous important properties, the most important of them being finiteness of the number of Laurent terms for  $f$ . A parallel notion can be defined for singularities of linear systems, but special care has to be exercised because of the multivaluedness of their solutions.

**Definition 16.1.** A vector or matrix function  $X(t)$ , eventually ramified at the origin, is said to be of *moderate growth* there if its norm grows at most polynomially in  $|t|^{-1}$  as  $t$  tends to the origin in any sector  $\alpha < \mathrm{Arg} t < \beta$  of

opening less than  $2\pi$ .

$$\|X(t)\| \leq C|t|^{-d}, \quad \text{as } |t| \rightarrow 0^+, \quad \alpha < \text{Arg } t < \beta, \quad (16.2)$$

for some finite  $d$  and  $C$  (which a priori may depend on the sector).

**Definition 16.2.** A singular point of a linear system is called *regular*, if some (hence any) fundamental matrix solution  $X(t)$  of the system has moderate growth at this point.

Differentiating the formula  $dX = \Omega X$ , we see that all derivatives of components of a fundamental solution also grow moderately at a regular singularity, since the meromorphic matrix form has at worst a pole at the singular point. This observation also remains valid for the higher derivatives of any finite order.

**Remark 16.3.** This terminology is counterintuitive, since “regular” does not mean “nonsingular”. However, it is too firmly established to replace the adjective “regular” by “tame” or “moderate” which would be less confusing.

**Lemma 16.4.** *For a regular singularity, the inverse  $X^{-1}(t)$  of any fundamental solution also grows moderately.*

**Proof.** From the monodromy property (16.1), the determinant  $h(t) = \det X(t)$  of any solution, is ramified over the origin:

$$\Delta h(t) = \mu h(t), \quad \mu = \det M \in \mathbb{C}^*.$$

The function  $t^{-\lambda}h(t)$ ,  $\lambda = (2\pi i)^{-1} \ln \mu$ , is therefore single-valued, not identically zero and growing moderately as  $t \rightarrow 0$ . Hence it must have a zero or pole of some finite order  $k \in \mathbb{Z}$ ,

$$\det X(t) = t^{k-\lambda}u(t), \quad u \in \mathcal{O}(\mathbb{C}, 0), \quad u(0) \neq 0.$$

Therefore the reciprocal  $1/h(t)$  is a function of moderate growth. The inverse  $X^{-1}$  can be expressed as  $(\det X)^{-1}$  times the adjugate matrix formed by all  $(n-1) \times (n-1)$ -minors of  $X(t)$ . Hence  $X^{-1}(t)$  also grows moderately.  $\square$

**Corollary 16.5.** *Let  $X(t)$  be a monodromic matrix function, such that  $\Delta X(t) = X(t)M$  for some nondegenerate matrix  $M$ . If  $X(t)$  has moderate growth, then the “logarithmic derivative”  $\Omega = dX \cdot X^{-1}$  is a meromorphic matrix 1-form.*

**Proof of the corollary.** The form  $\Omega$  is single-valued in the punctured neighborhood of the singular point:  $\Delta \Omega = dX \cdot MM^{-1}X^{-1} = \Omega$ . Because of the moderate growth,  $\Omega$  has at worst a pole at this point.  $\square$

**Lemma 16.6.** *If the homogeneous linear system (15.3) is regular at the origin and  $b(t)$  is a vector function of moderate growth at  $t = 0$ , then solutions of the nonhomogeneous system  $\dot{x} = A(t)x + b(t)$  also have moderate growth.*

**Proof.** This follows from the explicit formula (15.4).  $\square$

Meromorphic classification of regular singularities is very simple. Recall that cyclic matrix groups are isomorphic if their generators  $M, M'$  are conjugated by an invertible matrix,  $M' = C^{-1}MC$ . Isomorphism of the monodromy groups is a necessary condition of any gauge equivalence (Problem 15.9). For meromorphic gauge equivalence there are no other obstructions.

**Theorem 16.7** (meromorphic classification of regular singularities). *Any two regular singularities with the same monodromy are meromorphically gauge equivalent to each other.*

*In particular, any regular singularity is meromorphically equivalent to an Euler system.*

**Proof.** Without loss of generality we can find two fundamental matrix solutions  $X(t)$  and  $X'(t)$  for the two systems, which have *the same* monodromy matrix  $M \in \mathrm{GL}(n, \mathbb{C})$ :

$$\Delta X(t) = X(t)M, \quad \Delta X'(t) = X'(t)M.$$

Then the matrix ratio  $H(t) = X'(t)X^{-1}(t)$  is single-valued in the punctured neighborhood of the singular point, since

$$\Delta H = X'M \cdot M^{-1}X^{-1} = H.$$

Since  $H$  has (together with  $X', X$  and  $X^{-1}$ ) moderate growth, we conclude that  $H$  is a meromorphic matrix function, holomorphically invertible everywhere outside the singular point. By construction,  $H$  as a gauge map conjugates  $X$  with  $X' = HX$ .

Any monodromy matrix  $M$  has a matrix logarithm, thus there exists a complex matrix  $A$  such that  $\exp 2\pi i A = M$ . The corresponding Euler system  $dX = AX$  with the fundamental matrix solution  $X(t) = t^A$  has an arbitrary specified monodromy (Exercise 15.7).  $\square$

The explicit formula (15.13) for solutions of the Euler system implies the following corollary.

**Corollary 16.8.** *Any fundamental matrix solution of a linear system with a regular singularity at the origin, can be represented as*

$$X(t) = H(t)t^A, \quad H \in \mathrm{GL}(n, \mathcal{M}(\mathbb{C}, 0)), \quad A \in \mathrm{Mat}(n, \mathbb{C}) \quad (16.3)$$

*with some constant matrix  $A$  and meromorphic invertible matrix function (germ)  $H(t)$ .*  $\square$

**16B. Fuchsian singularities.** The problem of detecting regular singularities is in general rather difficult. For instance, Exercise 16.3 shows that no necessary condition of regularity can be given in terms of the Poincaré rank. However, there exists a simple *sufficient* condition of regularity.

**Definition 16.9.** A singularity is called *Fuchsian*, if its Pfaffian matrix has a simple pole, i.e., if its Poincaré rank  $r$  is equal to zero:

$$\Omega = (A_0 + A_1 t + \cdots) t^{-1} dt, \quad A_0, A_1, \cdots \in \text{Mat}(n, \mathbb{C}).$$

The matrix coefficient  $A_0$  before the term  $t^{-1}$  is called the *residue* of the Fuchsian singularity.

**Theorem 16.10** (L. Sauvage, 1886). *Any Fuchsian singularity is regular.*

**Proof.** In the logarithmic chart  $z = \ln t$  the Fuchsian system defined by a matrix 1-form  $\Omega = A(t) \cdot t^{-1} dt$  with the first order pole, becomes the linear system defined in some “sufficiently left” half-plane  $\{\text{Re } z < -B\}$ ,  $B \gg 0$  by a *bounded*  $2\pi i$ -periodic matrix 1-form  $\Omega' = A(\exp z) dz$ .

By the Gronwall inequality (Lemma 15.5), in any horizontal semi-strip  $\{\alpha < \text{Im } z < \beta, \text{Re } z < -B\}$  the norm of the fundamental matrix solution  $\|X(z)\|$  grows no faster than  $\|X(a)\| \cdot \exp K|z - a|$ , where  $a$  is a point on the right boundary of the strip and  $K = \sup \|A(z)\| < +\infty$ . Since the semi-strip is horizontal,  $|z - a| \leq |\beta - \alpha| + |\text{Re } z - B|$  on it. Combining these estimates and returning to the initial chart  $t = \exp z$ , we obtain the bound  $\|X(t)\| \leq \text{const } |t|^{-K}$  in the sector bounded by the rays  $\text{Arg } t = \alpha$  and  $\text{Arg } t = \beta$ .  $\square$

**Corollary 16.11.** *Any Fuchsian singularity is meromorphically gauge equivalent to an Euler system.*  $\square$

However, it would be wrong to assume that a Fuchsian system with the residue matrix  $A_0$  is always meromorphically equivalent to the Euler system  $t\dot{x} = A_0 x$  with the same matrix  $A_0$  (cf. with Problem 16.6). In the next several subsections we establish a polynomial integrable normal form for the local *holomorphic* classification of Fuchsian systems and prove its integrability, computing explicitly the fundamental solution and the monodromy.

**16C. Formal classification of Fuchsian singularities.** The first step in the local holomorphic classification of Fuchsian singularities consists of studying *formal equivalence*. Recall that two singularities  $\Omega, \Omega'$  are formally (gauge) equivalent, if there exists a *formal gauge transformation* defined by a formal series  $H \in \text{GL}(n, \mathbb{C}[[t]])$  such that the identity (15.10) holds on the level of formal power series.

As was observed by V. I. Arnold, the formal classification of Fuchsian singularities of linear systems can be reduced to the formal classification of *nonlinear* vector fields. Indeed, consider a system of linear equations

$$\dot{x} = t^{-1}(A_0 + tA_1 + t^2A_2 + \cdots)x,$$

and the corresponding meromorphic vector field (15.11) with  $r = 0$  in  $(\mathbb{C}, 0) \times \mathbb{C}^n$ . This analytic field is associated with the system of holomorphic *nonlinear* ordinary differential equations

$$\begin{aligned} \dot{x} &= A_0x + tA_1x + \cdots, \\ \dot{t} &= t, \end{aligned} \tag{16.4}$$

having an isolated singular point at the origin  $(0, 0)$ .

The linearization matrix that is block diagonal with two blocks, one being the residue matrix  $A_0$  of size  $n \times n$  and another  $1 \times 1$ -block consisting of the single entry 1. Without loss of generality we can assume that the matrix  $A_0$  is already in the upper-triangular Jordan normal form; its eigenvalues will be denoted  $\lambda_1, \dots, \lambda_n$ .

By the Poincaré–Dulac theorem, after an appropriate formal transformation one can remove from the system (16.4) all nonresonant terms. Yet the system (16.4), linear in all variables but one, has its specifics. On one hand, only the formal transformations from  $\text{Diff}[[\mathbb{C}^{n+1}, 0]]$  preserving the  $t$ -variable and linear in  $x$ -variables, are allowed by definition of the formal gauge equivalence. On the other hand, all resonant monomials are linear in  $x_1, \dots, x_n$  and have the form  $t^k x_j \frac{\partial}{\partial x_i}$ . Thus the only resonances between the eigenvalues  $\lambda_1, \dots, \lambda_n, 1$  that can prevent these monomials to be eliminated from (16.4), should have the form  $\lambda_i = \lambda_j + k$  with  $k \in \mathbb{Z}_+$ ; all other eventual resonances correspond to monomials that do not appear in (16.4) from the outset.

**Definition 16.12.** A Fuchsian singularity with the residue matrix  $A_0$  is *resonant*, if there are two eigenvalues of  $A_0$  that differ by a natural number. Otherwise the Fuchsian singularity is *nonresonant*.

In the resonant case one can immediately describe all resonant monomials linear in  $x$ . If  $A(t) = \sum_{k=0}^{\infty} t^k A_k$  is the matrix function containing only monomials resonant in the sense of Poincaré–Dulac, then *the matrix coefficient  $A_k$  may have nonzero entry at the  $(i, j)$ th position only if  $\lambda_i - \lambda_j = k$* . If the eigenvalues are arranged in the nonincreasing order in the sense of the partial order (11.3),

$$\lambda_i > \lambda_j \text{ in the sense (11.3)} \implies i < j \quad \forall i, j, \tag{16.5}$$

then the matrix  $A(t)$  is upper-triangular.

This condition formulated in terms of matrix elements, can be reformulated in terms of commutation of special matrices, i.e., as identity in  $\text{GL}(n, \mathbb{C})$ . Denote by  $\Lambda = \text{diag}\{\lambda_1, \dots, \lambda_n\}$  the diagonal part of the residue matrix  $A_0$ . For any constant matrix  $C$  the conjugacy  $C \mapsto t^A C t^{-A}$  by the power matrix function  $t^A$  multiplies  $(i, j)$ th element of  $C$  by  $t^{\lambda_i - \lambda_j}$ . Therefore the resonant terms  $A_k t^k$  can be described via their commutator with  $t^A$  as follows:

$$t^A A_k t^{-A} = t^k A_k, \quad k = 1, 2, \dots \quad (16.6)$$

**Definition 16.13.** A linear system of equations

$$\dot{x} = t^{-1}(A_0 + tA_1 + \dots + t^k A_k + \dots)x, \quad A_k \in \text{Mat}(n, \mathbb{C}), \quad (16.7)$$

with the residue matrix  $A_0$  is said to be in the *Poincaré–Dulac–Levelt normal form*, if

- (1) the residue matrix  $A_0$  is in the upper-triangular block diagonal Jordan form with the diagonal part  $\Lambda = \text{diag}\{\lambda_1, \dots, \lambda_n\}$ ,
- (2) the eigenvalues are enumerated in the nonincreasing order: if  $\lambda_i - \lambda_j = k \in \mathbb{N}$ , then  $i < j$ , i.e.,  $\lambda_j$ , follows after  $\lambda_i$ ,
- (3) the higher order matrix coefficients  $A_k$  satisfy the condition (16.6).

**Remark 16.14.** It is convenient to arrange the eigenvalues  $\lambda_1, \dots, \lambda_n$  in such a way that all eigenvalues with integer differences stay together and form a *resonant group* (the order of different, “incomparable” resonant group is not essential). If inside each group the eigenvalues are arranged in the decreasing order, then the matrix  $A(t)$  will have block-diagonal form with upper-triangular blocks corresponding to each resonant group. Such an arrangement will be convenient for reasoning.

Note also that the condition (16.6) for systems in the normal form is automatically satisfied also for  $k = 0$ : the matrix in the Jordan form commutes with its diagonal part. The requirement that  $A_0$  is (nonstrictly) upper-triangular is explicitly stated in Definition 16.13.

In the nonresonant case the Poincaré–Dulac–Levelt form is especially simple: it must be an Euler system with all  $A_k$  absent for  $k \geq 1$ . As there can be only finitely many differences between the eigenvalues, the Poincaré–Dulac–Levelt normal form is necessarily *polynomial*.

**Theorem 16.15** (Poincaré–Dulac theorem for Fuchsian singularities). *A Fuchsian singularity is formally equivalent to an upper-triangular system in the Poincaré–Dulac–Levelt normal form (16.7)–(16.6).*

*In particular, a Fuchsian system with a nonresonant residue matrix  $A_0$  is formally equivalent to the Euler system  $t\dot{x} = A_0 x$ .*

The proof of this theorem follows immediately from the Poincaré–Dulac Theorem 4.10. Indeed, Definition 16.13 is specifically designed so that the normal form contains all resonant terms and only them. All other (nonresonant) monomials can be eliminated from the system (16.4).

It remains only to check whether the resulting formal transformation will be linear in  $x_i$  and preserving the  $t$ -coordinate identically. This can be seen by inspection of the Poincaré–Dulac method: the normalizing map is constructed as an infinite composition of polynomial maps, each preserving the  $t$ -coordinate and linear in the  $x$ -coordinates, since only monomials of this form may need to be eliminated on each step.

However, the direct proof, largely parallel to the proof of Theorem 4.10, is shorter.

**Direct proof of the theorem.** To remove nonresonant terms of order  $k-1$  from the Fuchsian system whose matrix  $A(t) = t^{-1} \sum_{j \geq 0} t^j A_j$  has all lower order terms already normalized, consider a gauge equivalence with the conjugacy matrix  $H(t) = E + t^k H_k$ , whose inverse is  $H^{-1}(t) = E - t^k H_k + \dots$ . The transformed system will have the terms of order  $(k-1)$  as follows:

$$\begin{aligned} A'(t) &= kt^{k-1} H_k + t^{-1} (E + t^k H_k) A(t) (E - t^k H_k + \dots) \\ &= A(t) + t^{k-1} (kH_k + H_k A_0 - A_0 H_k) + \dots \end{aligned}$$

This computation shows that all matrix coefficients  $A'_0, \dots, A'_{k-1}$  of  $A'(t)$  will remain the same as the matrix coefficients of  $A(t)$ , while the last matrix coefficient  $A'_k$  can be modified by subtracting (or adding) any matrix  $B$  representable as  $kH + [H, A_0]$  for some  $H \in \text{Mat}(n, \mathbb{C})$ .

The operator of twisted commutation  $\mathbf{T}_k = k + \text{ad}_{A_0}: \mathcal{D}_1 \rightarrow \mathcal{D}_1$  on the space  $\mathcal{D}_1$  of linear vector fields (matrices) is lower triangular<sup>1</sup> in the basis  $\{x_i \frac{\partial}{\partial x_j} : 1 \leq i, j \leq n\}$  by Lemma 4.5 with the eigenvalues  $\lambda_i - \lambda_j - k$  on the diagonal. All nonresonant vector monomials  $x_i \frac{\partial}{\partial x_j}$  belong to the image of  $\mathbf{T}_k$  and hence can be eliminated, as explained in §4C.

In other words, the matrices  $A'_k$  can be brought into the resonant normal form containing nonzero entries only on  $(i, j)$  such that  $\lambda_i - \lambda_j = k$ . This entails the condition  $t^A A'_k t^{-A} = t^k A'_k$ . The process continues further by induction in  $k$ .  $\square$

**16D. Holomorphic classification of Fuchsian singularities.** As we have seen before, convergence of formal normalizing transformations for arbitrary nonlinear vector fields can be a rather delicate issue. However, for Fuchsian systems the situation is ideal.

**Theorem 16.16** (holomorphic classification of Fuchsian singularities). *Any formal gauge transformation conjugating two Fuchsian singularities, always converges.*

<sup>1</sup>Triangularity occurs with respect to the order of vector monomials chosen as the Lemma 4.5, regardless of the order of the variables  $x_1, \dots, x_n$  themselves.

In particular, any Fuchsian singularity is locally holomorphically equivalent to a polynomial Fuchsian system in the upper-triangular normal form (16.7)–(16.6). A nonresonant Fuchsian system is holomorphically equivalent to an Euler system.

The proof of this result can be obtained by several arguments. First, one can modify the proof of the Poincaré normalization Theorem 5.5 to show that the series converges; this is possible since all nonzero “small” denominators  $\lambda_i - \lambda_j - k$  are in fact bounded away from zero, exactly as in the Poincaré domain. However, there is an alternative simple proof avoiding all technical difficulties.

We start with a lemma concerning convergence of *formally meromorphic* solutions of Fuchsian systems. By definition, a formally meromorphic solution of a linear system (15.2) is a formal vector Laurent series

$$x(t) = \sum_{t=-d}^{+\infty} t^k x_k, \quad x_{-d}, \dots, x_0, x_1, \dots \in \mathbb{C}^n, \quad (16.8)$$

satisfying formally the equation (15.2).

**Lemma 16.17.** *Any formal meromorphic solution of a regular system is convergent and hence truly meromorphic.*

**Proof.** The property of having only convergent formally meromorphic solutions, is obviously invariant by (truly) meromorphic gauge equivalence of linear systems. As any regular system is meromorphically equivalent to an Euler system (Theorem 16.7), the assertion of the lemma is sufficient to prove only in this particular case.

For an Euler system  $t\dot{x} = Ax$ ,  $A \in \text{Mat}(n, \mathbb{C})$ , any formal solution (16.8) after substitution gives an infinite number of conditions

$$kx_k = Ax_k, \quad k = -d, \dots, 0, 1, \dots$$

Each of these conditions means that the vector coefficient  $x_k$  must be either zero or an eigenvector of  $A$  with the eigenvalue  $k \in \mathbb{Z}$ . But as soon as  $|k|$  exceeds the spectral radius of  $A$ , the second variant becomes impossible and hence all formal meromorphic solutions of the Euler system must be Laurent (vector) polynomials, thus converging.  $\square$

**Proof of Theorem 16.16.** Let  $H(t)$  be a formal matrix Taylor series conjugating two Fuchsian singularities  $\Omega_i = A_i(t) t^{-1} dt$ ,  $i = 1, 2$ . By (15.10), it means that

$$t^{-1}A_2 = \dot{H} \cdot H^{-1} + t^{-1}HA_1H^{-1},$$

implying the “matrix differential equation” for the matrix function  $H(t)$ ,

$$t\dot{H} = A_1H - HA_2.$$

This is *not* the equation in the form (15.3) with respect to the unknown matrix function  $H$ , since both left and right matrix multiplication occurs in the right hand side of this equation. However, it can be expanded to a *system* of  $n^2$  linear ordinary differential equations with respect to all  $n^2$  *entries* of the matrix  $H$ . The coefficients of this large  $(n^2 \times n^2)$ -system are picked from among the entries of  $t^{-1}A_i(t)$  and hence exhibit at most a simple pole at the origin.

All this means that  $H(t)$  is a formal vector solution to a Fuchsian system of order  $n^2$ . By Lemma 16.17, it converges.  $\square$

**16E. Integrability of the normal form.** Similarly to the nonlinear resonant Poincaré–Dulac normal forms, the Poincaré–Dulac–Levelt form is integrable even in the resonant case. This allows us to compute explicitly the corresponding monodromy operator.

Consider the matrix polynomial  $A(t) = A_0 + A_1t + A_2t^2 + \cdots + A_d t^d \in \text{Mat}(n, \mathbb{C}[t])$  in the Poincaré–Dulac–Levelt normal form, i.e., with the matrix coefficients  $A_k$  satisfying the conditions (16.6). The constant matrix difference

$$I = A(1) - \Lambda = (A_0 - \Lambda) + A_1 + \cdots + A_d, \quad (16.9)$$

is called the *characteristic matrix* of the corresponding Poincaré–Dulac–Levelt normal form.

The characteristic matrix  $I$  is nilpotent. Indeed, by Remark 16.14 all matrices  $A_1, \dots, A_d$  are strictly upper-triangular, and so is  $A_0 - \Lambda$ . Thus  $I$  is a strictly upper-triangular matrix involving contributions from both off-diagonal terms of the Jordan form of the residue  $A_0$  and also from the higher order terms of  $A(t)$ . Notice that in general  $\Lambda$  and  $I$  *do not commute*.

The characteristic matrix  $I$  allows us to write explicitly the fundamental matrix solution of a linear system in the normal form.

**Lemma 16.18.** *The system in the Poincaré–Dulac–Levelt normal form with the characteristic matrix  $I$  and the diagonal part of the residue  $\Lambda$  admits the fundamental matrix solution*

$$X(t) = t^\Lambda t^I. \quad (16.10)$$

**Proof.** Direct computation yields

$$\begin{aligned} t\dot{X}X^{-1} &= \Lambda + t^\Lambda I t^{-\Lambda} = t^\Lambda (\Lambda + I) t^{-\Lambda} \\ &= t^\Lambda (\Lambda + A_0 - \Lambda + A_1 + \cdots + A_d) t^{-\Lambda} \\ &= (\Lambda + A_0 - \Lambda) + tA_1 + \cdots + t^d A_d \\ &= A(t) \end{aligned}$$

by virtue of (16.6).  $\square$

If the matrices  $t^I$  and  $t^A$  were commuting, the monodromy of the system would be equal to the product  $\exp(2\pi iA)\exp(2\pi iI)$  (in any order). It turns out that the formula still holds even if  $[t^I, t^A] \neq 0$ .

**Corollary 16.19.** *The monodromy matrix  $M$  of the Poincaré–Levelt normal form is the product of two commuting matrices,*

$$M = \exp(2\pi iA)\exp(2\pi iI) = \exp(2\pi iI)\exp(2\pi iA). \quad (16.11)$$

**Proof.** Recall that a root subspace of an operator  $A_0$  corresponding to an eigenvalue  $\lambda$  is the maximal invariant subspace in  $\mathbb{C}^n$ , on which  $A_0 - \lambda E$  is nilpotent.

The space  $\mathbb{C}^n$  is the direct sum of *resonant subspaces*: by definition, each such subspace is the union of the *root subspaces* of all eigenvalues whose difference is an integer number. By construction, each resonant subspace is invariant by  $A_0$ . The conditions (16.6) guarantee also that the resonant space is invariant by all higher matrix coefficients  $A_k$ ,  $k = 1, 2, \dots$ .

The exponent of the diagonal term

$$\exp(2\pi iA) = \text{diag}\{\exp 2\pi i\lambda_1, \dots, \exp 2\pi i\lambda_n\}$$

is a *scalar* matrix on each resonant subspace of  $A$ , because all eigenvalues corresponding to this subspace have integer differences. Hence on each resonant subspace  $\exp(2\pi iA)$  commutes with  $I$ , thus also with  $t^I$  and  $\exp(2\pi iI)$ . Ultimately the monodromy operator  $\Delta$  around the singularity can be expressed as follows:

$$\begin{aligned} \Delta X(t) &= t^A \exp(2\pi iA) t^I \exp(2\pi iI) \\ &= t^A t^I \exp(2\pi iA) \exp(2\pi iI) \\ &= X(t)M, \end{aligned}$$

where  $M$  is given by the commuting product (16.11).  $\square$

For a nilpotent matrix  $I$  the matrix power  $t^I = \exp(\ln t I)$  is a matrix polynomial in  $\ln t$  of degree  $\leq n$ , hence Lemma 16.18 indeed yields a solution of the system in a closed form. Yet the true power of this result is a description of *invariant subspaces*, coordinate subspaces in  $\mathbb{C}^n$  of different dimensions, which are invariant by the flow of the Fuchsian system (15.2).

**Corollary 16.20.** *Eigenvalues  $\nu_j$  of the monodromy operator of a Fuchsian singular point may be put in one-to-one correspondence with the eigenvalues  $\lambda_j$  of the residue matrix in such a way that  $\nu_j = e^{2\pi i\lambda_j}$ .*

**Proof.** This is an immediate consequence of Lemma 16.18. It may be checked directly for the fundamental matrix (16.10). Choice of another fundamental matrix results in conjugacy of the monodromy operator, hence, leaves the eigenvalues unchanged.  $\square$