*Chapter III*

# Local and global theory of linear systems

Analysis of holomorphic vector fields and analytic foliations beyond the local theory exposed in Chapter I, is very difficult in more than two dimensions. Perhaps the only case where such a study is possible, both locally and globally, is that of (nonautonomous) *linear systems*. These systems exist on a rather special type of holomorphic manifolds, *holomorphic vector bundles*. The latter are "locally cylindric manifolds" made of cylinders (Cartesian products) $U \times \mathbb{C}^n$, $U \subseteq \mathbb{C}$ in the same way the manifolds are made of locally Euclidean charts. In this section we develop local and global theory of linear systems and their singularities.

## 15. General facts about linear systems

**15A. Linear differential equations: Pfaffian, ordinary, matrix.** Let $T$ be a Riemann surface, a complex one-dimensional (connected) manifold, which will play the role of the "complex time axis". The particular cases most important for the following are an arbitrarily small neighborhood of a point (punctured or not), subdomains of the complex line $\mathbb{C}$ and the Riemann sphere (the projective line $\mathbb{P}^1$ denoted for brevity by $\mathbb{P}$). We completely ignore in this book the linear systems defined over Riemann surfaces of positive genus.

Let $n$ be a natural number and $\omega_{ij} \in \Lambda^1(T)$, $i,j = 1,\dots,n$ a collection of $n^2$ holomorphic differential forms on $T$, arranged as an $n \times n$*-matrix* 1*-form*

$$\Omega = \begin{pmatrix} \omega_{11} & \cdots & \omega_{1n} \\ \vdots & \ddots & \vdots \\ \omega_{n1} & \cdots & \omega_{nn} \end{pmatrix} \in \operatorname{Mat}(n, \Lambda^1(T)).$$

Consider the complex $n$-space $\mathbb{C}^n$ equipped with the coordinates $x = (x_1,\dots,x_n)$ and the Cartesian product $T \times \mathbb{C}^n$. The one-dimensional distribution on $T \times \mathbb{C}^n$ defined by the common null space of the $n$ holomorphic 1-forms $\theta_i = dx_i - \sum_{j=1}^n \omega_{ij}x_j \in \Lambda^1(T \times \mathbb{C}^n)$, $i = 1,\dots,n$, defines a holomorphic foliation. Its leaves, considered as graphs of holomorphic vector functions $x(\cdot)\colon T \to \mathbb{C}^n$, are *solutions* of the *system of linear Pfaffian differential equations*

$$dx = \Omega x, \qquad \text{or, after expansion,} \qquad dx_i = \sum_{j=1}^n \omega_{ij}x_j. \tag{15.1}$$

Note that in general linear systems have only *multivalued* holomorphic solutions, since the leaves may cross many times the "vertical" lines $\{t = \text{const}\}$ which will be called *fibers* (having in mind future generalizations of the theory).

If $\widetilde{U} \subseteq T$ is a chart on $T$ with a coordinate function $t\colon \widetilde{U} \to \mathbb{C}$ on it with the range $U = t(\widetilde{U})$, then the 1-forms $\omega_{ij}$ and the respective matrix $\Omega$ can be represented as

$$\omega_{ij} = a_{ij}(t)\,dt, \qquad \text{resp.,} \qquad \Omega = A(t)\,dt,$$

where $a_{ij}(t)$ are holomorphic functions on $U$ together forming the holomorphic matrix function $A(t) \in \operatorname{Mat}(n, \mathcal{O}(U))$. In the chart $t$ the system of Pfaffian equations (15.1) takes the form of a system of $n$ *ordinary* linear differential equations

$$\dot{x}(t) = A(t)x(t), \qquad t \in U \subseteq \mathbb{C}, \quad x = (x_1\dots,x_n)^\top \in \mathbb{C}^n. \tag{15.2}$$

Together with vector solutions of the equations (15.1) or (15.2), it is very useful to consider also their *matrix* solutions. While any rectangular matrix solution with $n$ rows can be considered, the most important is the case of *square* $n \times n$-matrices. To distinguish the matrix equation from the vector one, we will choose the capital letters, writing

$$\begin{array}{ccc} dX = \Omega X, & & \dot{X}(t) = A(t)X(t), \\ \Omega \in \operatorname{Mat}(n, \Lambda^1(T)), & \text{or} & A(t) \in \operatorname{Mat}(n, \mathcal{O}(T)), \\ \text{(Pfaffian)}, & & \text{(ordinary)}, \end{array} \tag{15.3}$$

$$X = X(t) \in \operatorname{Mat}(n, \mathcal{O}(t)), \quad \Omega = A(t)\,dt, \quad t \in T.$$

Such matrices represent $n$-tuples of vector solutions of (15.2) or (15.1).

**15B. Fundamental solutions.** In general not all solutions of differential equations can be continued over all paths. Linear systems are exceptionally well behaved in this respect.

**Proposition 15.1.** *Any solution of a differential system* (15.1) *can be continued as an analytic vector function along any simple path* $\gamma \subset T$. *The result of this continuation is a linear map* $\Delta_\gamma$ *between the fibers* $\tau_a$ *and* $\tau_b$ *at the endpoints* $a, b \in T$ *of the path* $\gamma$.

**Proof.** The zero solution obviously can be continued along any path, thus all sufficiently close solutions can also be continued. Yet because of the linearity, any solution admits continuation.

More precise arguments run as follow. The *null leaf*, the curve $T \times \{0\} \subset T \times \mathbb{C}^n$, is always the leaf of *any* foliation defined by a linear system (15.1). This curve is transversal to each fiber $\tau_a = \{a\} \times \mathbb{C}^n \subset T \times \mathbb{C}^n$.

Let $\gamma \subset T$ be any path connecting two points $a, b \in T$. Then for any foliation $\mathcal{F}$ defined by a system (15.1) the holonomy (correspondence) map $\Delta_\gamma \colon (\tau_a, a) \to (\tau_b, b)$ is always defined between *sufficiently small* neighborhoods of the points $a$ and $b$ on the respective cross-sections $\tau_a, \tau_b$, as explained in §2**C**. Yet because of the linearity of the system, the holonomy map is in fact *linear* and hence can be defined between the entire transversals $\tau_a, \tau_b \cong \mathbb{C}^n$.

Indeed, let $x'(t), x''(t)$ be solutions of (15.1) corresponding to the initial conditions $v', v'' \in \tau_a$ and small enough so that both their graphs and the graph of their sum $x(t) = x'(t) + x''(t)$ belong to a sufficiently small tubular neighborhood of the curve $\gamma \subset T \times \{0\} \subset T \times \mathbb{C}^n$. Then the sum $x(t)$ also satisfies (15.1)

$$dx = d(x' + x'') = dx' + dx'' = \Omega x' + \Omega x'' = \Omega(x' + x'') = \Omega x,$$

with the initial condition $v = v' + v''$ and takes the terminal value $\Delta_\gamma(v) = \Delta_\gamma(v') + \Delta_\gamma(v'')$. A similar argument shows also that $\Delta_\gamma(cv) = c\Delta_\gamma(v)$ for all sufficiently small $v$ and $cv$. □

From now on we will always consider the holonomy maps $\Delta_\gamma$ as globally defined (automatically invertible) linear maps between different copies of $\mathbb{C}^n$ if $a \neq b$ or linear self-maps from $\mathrm{GL}(n, \mathbb{C})$ if $a = b$ and the path $\gamma$ is a closed loop. As follows from the general properties of the holonomy, the map $\Delta_\gamma$ depends only on the homotopy class of the path $\gamma$ with the fixed endpoints. In particular, if $T$ is simply connected, then the correspondence map $\Delta_a^b$ is well defined for any two endpoints $a, b \in T$. In this case solutions of the linear system obviously form a *linear space* over $\mathbb{C}$.

**Definition 15.2.** A tuple of solutions is called a *fundamental system of solutions* of the systems (15.1) or (15.2) on a simply connected base $T$, if it is a basis in the linear space of all such solutions. A *fundamental matrix solution* of the equation (15.3) is a holomorphic matrix function $X: T \mapsto \operatorname{Mat}(n, \mathbb{C})$ which is everywhere *nondegenerate*, $\det X(t) \neq 0$ for all $t \in T$.

The following basic result describes the structure of the linear space of solutions of a linear system.

**Theorem 15.3.** *Any linear system* (15.1) *of order $n$ over a simply connected Riemann surface $T$ admits an $n$-dimensional linear space of solutions. The "evaluation map" $x(\cdot) \mapsto x(a)$ assigning to any solution $x(\cdot)$ its value $x(a)$ is an isomorphism between this space and the vertical cross-section $\tau_a = \{a\} \times \mathbb{C}^n$ for any choice of the point $a \in T$.*

*Any $n$ solutions with linearly independent initial conditions in $\tau_a$ form a fundamental matrix solution defined on the entire surface $T$. Any two fundamental matrix solutions $X(t), X'(t)$, differ by a constant right matrix factor, $X'(t) = X(t)C$, where $C \in \operatorname{Mat}(n, \mathbb{C})$.*

**Proof.** Let $a \in T$ be a fixed base point. Then for any $s \in T$ the holonomy map $\Delta_a^s$ between the cross-sections $\tau_a = \{t = a\}$ and $\tau_s = \{t = s\}$ is a uniquely defined linear operator by Proposition 15.1, and for any initial value $v \in \tau_a$ the function

$$s \longmapsto x_v(s) = \Delta_a^s(v), \qquad s \in T,$$

is a globally defined solution to the linear system (15.1). The map $v \mapsto x_v(\cdot)$, inverse to the evaluation map $x(\cdot) \mapsto x(a)$, is a linear operator between two linear spaces: it is surjective by the existence part and injective by the uniqueness part of the assertion of Theorem 1.1.

Choosing any $n$ linearly independent initial values $v_1, \dots, v_n \in \tau_a$ and arranging them into the nondegenerate square matrix $V$, we may as before use the holonomy $\Delta_a^s$ to construct the matrix solution $s \mapsto X_V(s)$. This solution is automatically nondegenerate at every point, since all holonomy operators $\Delta_a^s$ are invertible.

To prove the last remaining assertion, consider the quotient $C(t) = X^{-1}(t)X'(t)$ of two fundamental matrix solutions for the same system (15.3). Differentiating this quotient, we obtain

$$\begin{aligned} dC = d(X^{-1}X') &= -X^{-1}\,dX \cdot X^{-1}X' + X^{-1}\,dX' \\ &= -X^{-1}\Omega X' + X^{-1}\Omega X' = 0, \end{aligned}$$

which means that this quotient is a constant invertible matrix $C$. □

**Remark 15.4.** An alternative proof of the fact that any solution of a linear system can be continued along any path, can be achieved by purely real arguments. We start with a general a priori growth rate bound characteristic for linear systems.

**Lemma 15.5** (Gronwall inequality)**.** *Let $A(\cdot)$ be a continuous matrix function on the real interval $t \in [t_0, t_1] \subset \mathbb{R}$ of explicitly bounded norm,*

$$\forall t \in [t_0, t_1] \qquad A(t) \in \operatorname{Mat}(n, \mathbb{C}), \qquad \|A(t)\| \leqslant c.$$

*Then any solution $x(t)$ of the linear system* (15.2) *satisfies the inequality*

$$\|x(t)\| \leqslant \|x(t_0)\| \exp(c\,|t - t_0|).$$

**Proof.** By the limit triangle inequality, $\frac{d}{dt}\|x(t)\| \leqslant \|\frac{d}{dt}x(t)\|$, therefore

$$\tfrac{d}{dt}\|x(t)\| \leqslant \|A(t)\|\,\|x(t)\| \leqslant c\,\|x(t)\|.$$

Therefore the logarithmic derivative $\frac{d}{dt}\ln\|x(t)\|$ is bounded by $c$ everywhere on $[t_0, t_1]$, so that its growth between $t_0$ and an arbitrary $t$ is no greater than $C\,|t - t_0|$. This immediately implies the inequality for the norm $\|x(t)\|$ itself. ☐

By the Gronwall inequality, any solution with the initial condition $x_0 \in \mathbb{R}^n$ cannot leave the compact set $K = [t_0, t_1] \times \{\|x\| \leqslant R'\} \subset \mathbb{R}^{1+n}$, $R' = \|x_0\|\, \exp(R\,|t_1 - t_0|)$, except for the right section $K \cap \{t_1\} \times \mathbb{R}^n$. On the other hand, by one of the fundamental theorems for real ordinary differential equations [**Arn92**], any solution beginning in any compact $K \subset \mathbb{R} \times \mathbb{R}^n$ in the “space-time” can be continued until it reaches the boundary of $K$. Together with the above argument, this implies that solutions of linear systems on real intervals are always globally defined.

One can use this real theorem to continue solutions along arbitrary parameterized curves in $T$. It remains to prove that these restricted solutions are in fact holomorphic on $T$ and prove (in the same way as before) that the results are independent of the choice of the curves in case the domain is simply connected.

**Remark 15.6** (variation of constants)**.** Solution of nonhomogeneous systems can be reduced to that of homogeneous systems using the method of *variation of constants.* If $X(t)$ is a fundamental matrix solution of the linear system $dX = \Omega X$, then a particular solution of the nonhomogeneous system $dY = \Omega Y + \Theta$, where $\Theta$ is a known matrix 1-form on $T$, is given by the formulas

$$Y(t) = X(t)C(t), \qquad dC = X^{-1}\Theta, \tag{15.4}$$

where solutions of the second equation can be found by immediate integration, $C = \int X^{-1}\Theta$, since any holomorphic 1-form on a simply connected Riemann surface $T$ is exact. Any other solution of the nonhomogeneous system can be obtained as the sum of the particular solution $Y(t)$ and a general solution of the homogeneous system.

**15C. Monodromy and holonomy.** If the Riemann surface $T$ is not simply connected, the leaves of the foliation $\mathcal{F}$ tangent to the distribution $dx - \Omega x = 0$ on $T \times \mathbb{C}^n$ in general are not graphs of vector functions: they may intersect the fibers $\tau_a = \{t = a\} \times \mathbb{C}^n \subset T \times \mathbb{C}^n$ by more than one point. In the classical language it is said that solutions of the system (15.3) are *multivalued* functions of $t$. Speaking geometrically, the foliation $\mathcal{F}$ may have a nontrivial *holonomy group*.

As it was defined in §2**C**, this group associates with any loop $\gamma \in \pi_1(T, a)$ on the null leaf $T \cong L_0 = \{x = 0\} \in \mathcal{F}$ a linear invertible self-map $\Delta_\gamma$ of the cross-section $\tau_a$. If a coordinate system is fixed on the section $\tau_a$, then $\Delta_\gamma$ becomes a square matrix denoted by $F_\gamma$. By construction, for any fundamental matrix solution $X(t)$ of (15.3) the result of its analytic continuation over the loop $\gamma$ is

$$\Delta_\gamma X(a) = F_\gamma X(a), \qquad F_\gamma \in \mathrm{GL}(\tau_a) \cong \mathrm{GL}(n, \mathbb{C}). \tag{15.5}$$

Note that the linear operators $F_\gamma$ depend on the choice of the base point $a$.

A different construction requires a choice of fundamental matrix solution. If $X(t)$ is such a solution, then the result of its analytic continuation along a loop $\gamma \in \pi_1(T, a)$ is another fundamental matrix solution. By Theorem 15.3, there exists a constant matrix $M = M_\gamma$, called the *monodromy matrix*, such that

$$\Delta_\gamma X(t) = X(t) \cdot M_\gamma, \qquad M_\gamma \in \mathrm{GL}(n, \mathbb{C}). \tag{15.6}$$

The monodromy matrices *do not depend on the choice of the base point* $a \in T$ in the following sense: the identity (15.6) holds for all points $t$ sufficiently close to $a$, if we identify the loops $\gamma \in \pi_1(T, t)$ for different base points $t$ sufficiently close to $a$. On the other hand, the monodromy matrices depend on the choice of a fundamental solution $X(t)$: choosing a different fundamental solution $X'(t) = X(t)C$ results in replacing $M_\gamma$ by $M'_\gamma = C^{-1} M_\gamma C$.

Both correspondences, the holonomy $\gamma \mapsto F_\gamma$ and the monodromy $\gamma \mapsto M_\gamma$, are linear representations of the fundamental group:

$$\forall \gamma_1, \gamma_2 \in \pi_1(T, a) \qquad M_{\gamma_1 \cdot \gamma_2} = M_{\gamma_2} M_{\gamma_1}, \qquad F_{\gamma_1 \cdot \gamma_2} = F_{\gamma_2} F_{\gamma_1}, \tag{15.7}$$

where $\gamma_1 \cdot \gamma_2$ is the composite loop circumscribing first $\gamma_1$ and then $\gamma_2$. The two representations are equivalent: as follows from their definitions, the monodromy matrices $M_\gamma$ *numerically coincide* with the holonomy matrices $F_\gamma$ for the standard choice of coordinates on $\mathbb{C}^n$ and a special choice of the fundamental solution $X(t)$, normalized by the condition $X(a) = E$. The image of these representations in $\mathrm{GL}(n, \mathbb{C})$ will be referred to as the *monodromy group* of the linear system (15.3).

**15D. Gauge transform and gauge equivalence.** The special structure of the phase space on which linear systems are defined, restricts the class of admissible transformations. Instead of arbitrary biholomorphisms of the Cartesian product $T \times \mathbb{C}^n$, only maps linear in the "vertical" ("linear") variables are allowed.

More precisely, consider two cylinders $S = T \times \mathbb{C}^n$ and $S' = T' \times \mathbb{C}^n$ over two Riemann surfaces $T$ and $T'$ respectively. Each cylinder is naturally equipped by the projection $\pi\colon S \to T$ (resp., $\pi'\colon S' \to T'$) on the base. A *gauge map*, or *gauge transform* between these two cylinders is a holomorphism $H\colon S \to S'$ which respects these projections and is linear on each fiber $\tau_a = \pi^{-1}(a) = \{a\} \times \mathbb{C}^n$, for any $a \in T$. This means that there exists a holomorphic map $h\colon T \to T'$ such that

$$\begin{gathered} \pi' \circ H = h \circ \pi, \qquad H|_{\tau_a}\colon \tau_a \to \tau_{h(a)} \quad \text{is linear,} \\ \tau_a = \pi^{-1}(a), \qquad \tau_{h(a)} = \pi'^{-1}(h(a)). \end{gathered} \tag{15.8}$$

In coordinates a gauge map takes the form

$$(t, x) \mapsto \bigl(h(t), H(t)x\bigr), \qquad H \in \mathrm{GL}(n, \mathcal{O}(T)), \tag{15.9}$$

where $H(\cdot)$ is a holomorphic matrix function ("linear change of the dependent variables"). If necessary, we will specify explicitly that the gauge transform is fibered over the base map $h$. A gauge map is invertible if and only if $h$ is a biholomorphism and $H(t)$ is invertible for any $t \in T$. In practice we will almost always consider cylinders over the same Riemann surface and use maps fibered over the identity map $h = \mathrm{id}$. The holomorphic invertible matrix function $H = H(t) \in \mathrm{GL}(n, \mathcal{O}(T))$ is called the *conjugacy matrix.*

Gauge equivalence naturally acts on linear systems defined on the respective cylinders. If $X(t)$ is a fundamental matrix solution to a system $dX = \Omega X$ and $H\colon (t, x) \mapsto \bigl(t, H(t)x\bigr)$ a gauge map, then the image $X'(t) = H(t)X(t)$ is a fundamental matrix solution to another linear system $dX' = \Omega' X'$. One can immediately see by expanding the expression for the derivative $dX' \cdot X'^{-1}$, that

$$\Omega' = dH \cdot H^{-1} + H \cdot \Omega \cdot H^{-1}. \tag{15.10}$$

Two linear systems of the same order defined on the same Riemann surface $T$, are said to be *gauge equivalent* (more precisely, holomorphically gauge equivalent) if they can be transformed into each other by an invertible gauge map.

Clearly, *gauge equivalent systems have isomorphic monodromy and holonomy groups.* The corresponding matrix representations are equivalent. If the two fundamental solutions used to compute the monodromy group are $X(t)$ and $X'(t) = H(t)X(t)$, then the monodromy matrices *coincide* identically. This explains why in many cases the monodromy matrices are more convenient to deal with than the holonomy operators. The holonomy groups for two gauge equivalent systems, if associated with the same base point $a \in T$, are linearly conjugate by the map $H(a) \in \mathrm{GL}(\tau_a)$.

**15E. Systems with isolated singularities.** A *linear system with singularities* over a Riemann surface $T$ is a singular holomorphic foliation $\mathcal{F}$ on $T \times \mathbb{C}^n$ which coincides with a foliation defined by some linear system (15.3) outside a “small” exceptional set. The exception, nonanalyticity locus $\Sigma$ of the matrix 1-form $\Omega$, is a subset of the complex one-dimensional base $T$. It is reasonable to assume that this set is discrete (zero-dimensional), so that $\mathcal{F}$ is defined on the complement $(T \smallsetminus \Sigma) \times \mathbb{C}^n$ to an analytic subset of complex codimension 1.

One can show that in order to be extendable to the complement of an analytic subset of codimension $\geqslant 2$, the matrix elements of the form must be *meromorphic*, i.e., have at worst a finite order pole at every point of the singular locus $\Sigma$; cf. with Problem 15.5. In this case, assuming that the singular point is at the origin $t = 0$, the foliation can be generated by a holomorphic vector field

$$F = t^{r+1}\frac{\partial}{\partial t} + \sum_{j=1}^{n} a_{ij}(t)x_j\frac{\partial}{\partial x_j}, \qquad r \in \mathbb{Z}_+, \quad (t,x) \in (\mathbb{C}^1, 0) \times \mathbb{C}^n. \tag{15.11}$$

One can see immediately that the singularities of the foliation after maximal extension are all *isolated* (apriori they should only form a locus of complex codimension $\geqslant 2$) and belong to the closure of the null leaf $T \times \{0\}$ which thus becomes a *common separatrix for all singularities.* This observation explains the special role that the null leaf plays in investigation of the linear systems.

**Definition 15.7.** A *linear system with singularities* on a Riemann surface $T$ is the singular holomorphic foliation defined by a Pfaffian system (15.3) with a *meromorphic* matrix 1-form $\Omega \in \operatorname{Mat}(n, \mathcal{M}(T))$ with the polar locus $\Sigma \subset T$. Points of this locus are called *singular points*, or simply *singularities* of the linear system.

Clearly, a linear system with singularities on $\Sigma$ is a holomorphic (nonsingular) linear system on the Riemann surface $T' = T \smallsetminus \Sigma$. Since $T'$ is not simply connected when $\Sigma \neq \varnothing$, the holonomy (monodromy) group of this restricted system is usually nontrivial.

**Definition 15.8.** The monodromy group of a linear system with singularities on the Riemann surface is the monodromy group of its restriction on $(T \smallsetminus \Sigma) \times \mathbb{C}^n$.

**Example 15.9** (Euler system)**.** A linear system with constant coefficients, $\Omega = A\,dt$, has no singularities on $\mathbb{C}$ but when considered on $\mathbb{P}$, it has a pole of second order at infinity: in the chart $w = 1/t$, it has the Pfaffian matrix $\Omega = -Aw^{-2}dw$.

A simplest nontrivial example of a linear system on $\mathbb{P}$ having the minimal number of *simple* poles, is the *Euler system*,

$$dX = \Omega X, \qquad \Omega = A\,t^{-1}dt, \qquad A \in \operatorname{Mat}(n, \mathbb{C}), \tag{15.12}$$

defined by a single constant matrix $A$ called the *residue*. The singular locus of the system (15.12) on $\mathbb{P}$ consists of two points $\{0, \infty\}$.

The Euler system can be immediately integrated. Consider the logarithmic chart $z = \ln t$ on the universal covering $\mathbb{C}$ of $\mathbb{P} \smallsetminus \Sigma$. In this chart (15.12) becomes a system with constant coefficients $\Omega = A\,dz$, whose fundamental solution is given by the matrix exponent. In the initial chart the exponential solution takes the form

$$X(t) = t^A = \exp(A \ln t), \qquad t \neq 0 \tag{15.13}$$

which is indeed ramified over $\Sigma$.

The fundamental group of $\mathbb{P} \smallsetminus \Sigma = \mathbb{C} \smallsetminus \{0\}$ is cyclic, generated by the loop $s \mapsto \exp 2\pi i s$, $s \in [0,1]$, around the origin. The monodromy matrix of the Euler equation, corresponding to the above constructed fundamental solution, can be easily computed:

$$M = \exp 2\pi i A \tag{15.14}$$

(going around the origin corresponds to choosing a different branch of the logarithm, shifted by $2\pi i$ from the initial one).

The integer index $r \geqslant 0$ determining the order of pole of the matrix $\Omega$ at a singular point, is called the *Poincaré rank* of the corresponding singularity. The holomorphic gauge transformations act in a natural way on meromorphic linear systems as well. Obviously, the Poincaré rank is invariant by the gauge equivalence.

**Remark 15.10.** Once the class of holomorphic linear linear systems is extended to the class of meromorphic linear systems (with singularities), it is natural to extend also the class of admissible gauge transformations, relaxing holomorphy of the matrix function $H(t)$ in (15.9) to meromorphy of $H(t)$ together with its inverse $H^{-1}(t)$.

However, the *meromorphic gauge equivalence* introduced this way, is too strong. In particular, any two systems with poles of first order (i.e., of Poincaré rank zero) are meromorphically gauge equivalent if and only if their monodromy groups are isomorphic, both locally and globally (Problem 16.2). On the other hand, the Poincaré rank is not necessarily preserved by meromorphic gauge transformations.

## Exercises and Problems for §15.

**Problem 15.1.** Prove directly, using Theorem 1.1, that for any point $a \in T$ there exists a small neighborhood $U_\alpha \subset T$ of $a$ such that the linear system (15.3) has a fundamental matrix solution $X_\alpha$ in $U$, normalized by the condition $X_\alpha(a) = E$.

**Exercise 15.2.** Assume that the covering $\{U_\alpha\}$, constructed in the previous problem, is finite. Prove that the constant matrix factors $C_{\alpha\beta} = X_\alpha^{-1} X_\beta$ defined on the pairwise intersections $U_{\alpha\beta} = U_\alpha \cap U_\beta$, satisfy the identities

$$C_{\alpha\beta} C_{\beta\alpha} = \mathrm{id}, \qquad C_{\alpha\beta} C_{\beta\gamma} C_{\gamma\alpha} = \mathrm{id} \tag{15.15}$$

on $U_\alpha \cap U_\beta$ and $U_\alpha \cap U_\beta \cap U_\gamma$ respectively (whenever the latter intersections are nonempty).

**Problem 15.3.** Consider a *simply connected* Riemann surface $T$ and its covering $U_\alpha$ by open domains such that all nonempty pairwise and triple intersections are connected.

Prove that for any collection of matrices $C_{\alpha\beta}$ satisfying the identities (15.15), one can find constant matrices $C_\alpha$ so that $C_{\alpha\beta} = C_\alpha^{-1} C_\beta$ whenever the intersection $U_\alpha \cap U_\beta$ is nonempty.

What happens if $T$ is not simply connected?

**Problem 15.4.** Derive from Exercise 15.2 and Problem 15.3 that any linear system on a simply connected Riemann surface admits a globally defined fundamental matrix solution.

**Problem 15.5.** Let $\mathcal{F}$ be a holomorphic foliation generated by a linear system $dx - \Omega x = 0$ on the cylinder $T \times \mathbb{C}^n$ outside the locus $\Sigma \times \mathbb{C}^n$, where $\Sigma \subset T$ is a discrete set.

Prove that this foliation extends as a singular holomorphic foliation with a singular locus of codimension $\geqslant 2$ on $T \times \mathbb{C}^n$ if and only if $\Omega$ has a finite order pole at every point of $\Sigma$.

**Problem 15.6.** Prove that any linear system on $\mathbb{P}$ with two simple poles is gauge equivalent to the Euler system (15.12).

**Exercise 15.7.** Prove that *any* nondegenerate matrix $M$ can be realized as the monodromy of an appropriate Euler system.

**Problem 15.8.** Let $A_1, \dots, A_m \in \operatorname{Mat}(n, \mathbb{C})$ be *commuting* constant matrices with $A_1 + \cdots + A_m = 0$, and $t_1, \dots, t_m \in \mathbb{C}$ different points.

Prove that the rational 1-form $\Omega = \sum_1^m A_j \, \frac{dt}{t - t_j}$ defines a singular linear system on $\mathbb{P}^1$. Describe the singular locus and the monodromy group of this system.

**Problem 15.9.** Prove that two holomorphically or meromorphically gauge equivalent linear systems have isomorphic monodromy groups.

**Problem 15.10** (Liouville–Ostrogradskii formula)**.** Let $X(t)$ be a meromorphic matrix function in a domain $U$ with $\det X \not\equiv 0$, and $\Omega = dX \cdot X^{-1}$ the meromorphic matrix 1-form (the “logarithmic derivative” of $X$). Prove that the scalar 1-form $\operatorname{tr} \Omega$ is the logarithmic derivative of $\det X$, i.e., it satisfies the identity $\operatorname{tr} \Omega = d(\det X) \cdot (\det X)^{-1}$ in $U$; cf. with (1.16).