Problem 7.5. Assume that the first k eigenvalues $\lambda_1, \dots, \lambda_k$ from the spectrum of a holomorphic vector field $F \in \mathcal{D}(\mathbb{C}^n, 0)$, $k \leqslant n$, are real, and the others are not.

Prove that the field F has a holomorphic k-dimensional invariant manifold tangent to the coordinate plane generated by the first k basis vectors.

8. Desingularization in the plane

Reasonably complete analysis of singular points of holomorphic vector fields using holomorphic normal forms and transformations by biholomorphisms, is possible under the assumption that the linear part is not very degenerate. The degenerate cases have to be treated by transformations that can alter the linear part. Such transformations, necessarily not holomorphically invertible, are known by the name *desingularization*, *resolution of singularities*, *sigma-process* or *blow-up*. Very roughly, the idea is to consider a holomorphic map $\pi\colon M \to (\mathbb{C}^2, 0)$ of a holomorphic surface (2-dimensional manifold) M that squeezes (blows down) a complex 1-dimensional curve $D \subset M$ to the single point $0 \in \mathbb{C}^2$, while being one-to-one between $M \smallsetminus D$ and $(\mathbb{C}^2, 0) \smallsetminus \{0\}$. The second circumstance allows us to pull back local objects (functions, curves, foliations, 1-forms, *etc.*) from $(\mathbb{C}^2, 0)$ to M and then extend them on D. These pullbacks are called desingularizations, or blow-ups of the initial objects; sometimes M is itself called the blow-up of (the neighborhood of) the point $0 \in \mathbb{C}^2$.

In this section we develop some basic algebraic geometry necessary to deal with desingularizations and introduce the notion of multiplicity of an isolated singularity of a foliation.

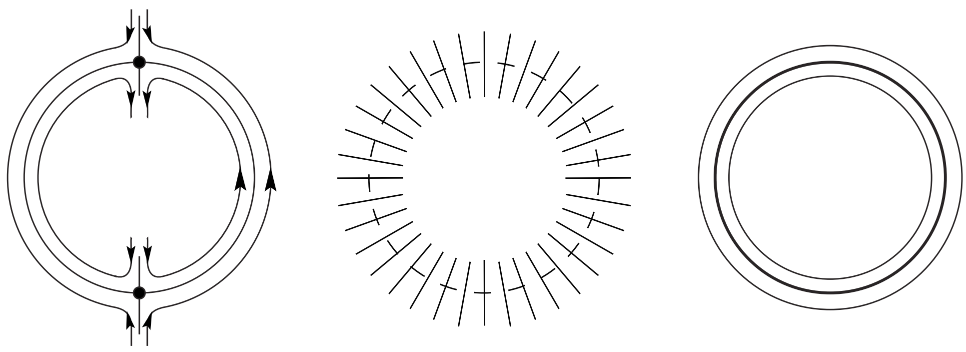
Using desingularization one can ultimately simplify singularities of holomorphic foliations in dimension 2. The main result of this section, the fundamental Desingularization Theorem 8.14 asserts that by a suitable blow-up any singular holomorphic foliation in a neighborhood of a singular point can be resolved into a singular foliation, defined in a neighborhood of a union $D = \bigcup_i D_i$ of one or more transversally intersecting holomorphic curves D_i, which has only *elementary* singularities on D.

8A. Polar blow-up. We start with a transcendental but geometrically more transparent construction in the real domain.

Definition 8.1. The *polar blow-down* is the map P of the real cylinder $C = \mathbb{R} \times \mathbb{S}^1 \to \mathbb{R}^2$ onto the plane $\mathbb{R}^2$,

$$P\colon (r, \varphi) \mapsto (r \cos \varphi, r \sin \varphi). \tag{8.1}$$

This map is a real analytic diffeomorphism between the open half-cylinder $C_+ = \{r > 0\} \subset C$ and the punctured plane $\mathbb{R}^2 \smallsetminus \{0\}$. The



(i) (ii) (iii)

Figure I.4. Trigonometric blow-up of a nonsingular (i) and singular (ii), (iii) foliations

image of the narrow band $C = (\mathbb{R}, 0) \times \mathbb{S}^1$ (cylinder) is a double covering of the small neighborhood of the origin $\{|x| < \varepsilon\}$ except for the central equator $S = \{r = 0\} \subset C$, also called the *exceptional divisor*. The latter is squeezed into one point, the origin $0 \in \mathbb{R}^2$.

The map P pulls back functions and differential 1-forms from $(\mathbb{R}^2, 0)$ on C (in noninvariant terms, passing to the polar coordinates and ignoring the inequality $r > 0$). However, the pullback $P^*\omega \in \Lambda^1(C)$ of any 1-form $\omega \in \Lambda^1(\mathbb{R}^2, 0)$ always has a nonisolated singularity on S. In the real analytic case one can always divide $P^*\omega$ by a suitable natural power r^ν so that the 1-form $\widetilde{\omega} = r^{-\nu} P^*\omega \in \Lambda^1(C)$ still remains real analytic but has only isolated singularities on S.

Consider now the singular foliation $\mathcal{F}$ defined by the Pfaffian equation $\{\omega = 0\}$ on $(\mathbb{R}^2, 0) \smallsetminus \{0\}$. As P is one-to-one outside the origin, $P^{-1}(\mathcal{F})$ is a foliation of $C \smallsetminus S$ with the Pfaffian equation $\{P^*\omega = 0\}$. Since r is nonvanishing outside S, the foliation $\widetilde{\mathcal{F}}$ can be defined by the Pfaffian equation $\{\widetilde{\omega} = 0\}$ which has only isolated singularities on S and thus extends $P^{-1}(\mathcal{F})$ as a singular foliation on C.

Definition 8.2. The line field defined by the Pfaffian distribution $r^{-\nu} P^*\omega = 0$ with isolated singularities and the corresponding singular foliation $\widetilde{\mathcal{F}}$ on C are called the *trigonometric blow-up* of the distribution $\omega = 0$ and the corresponding foliation $\mathcal{F}$ respectively.

Intuitively, the singular point is “stretched” to the central circle so that the complicated behavior of leaves near the singularity can be studied in different “parts” separately. The first examples show that sometimes the singularity may even disappear.

Example 8.3.

(i) The form $dx = 0$ defining a nonsingular foliation, after trigonometric blow-up becomes $\cos\varphi\, dr - r\sin\varphi\, d\varphi$ and has two isolated singular points $(0,0)$ and $(0,\pi)$ on $\mathbb{R}\times\mathbb{S}^1$. Both these points are nondegenerate saddles. The exceptional circle without these points is the leaf of the blow-up foliation.

(ii) The form $\omega = y\,dx - x\,dy$ defines the "radial" singular foliation on $\mathbb{R}^2$. The pullback $P^*\omega = -r^2\,d\varphi$, has a nonisolated singularity on $r = 0$, but after division the form $\widetilde{\omega} = r^{-2}P^*\omega = d\varphi$ defines the nonsingular "parallel" foliation $\{\varphi = \text{const}\}$. All leaves of this foliation cross the exceptional circle S transversally.

(iii) The form $x\,dx + y\,dy = \frac{1}{2}d(x^2+y^2)$ which defines the foliation of $\mathbb{R}^2$ by the circles $x^2 + y^2 = \text{const}$, pulls back as the line field $r\,dr = 0$ which after division also becomes a nonsingular form dr on C. The exceptional circle is a leaf of the blow-up foliation carrying no singular points.

The map P can be complexified and the above examples generalized. However, the complexification will also be a two-fold covering, which is not natural geometrically. Besides, using the trigonometric functions $\sin\varphi, \cos\varphi$ makes the corresponding formulas nonalgebraic.

There is an algebraic version of the map P, called the *sigma-process, monoidal transformation*, or simply the *blow-up* without the adjective trigonometric.

8B. Algebraic blow-up (σ-process). It is not so easy to construct a holomorphic 2-dimensional manifold M and a holomorphic map $\sigma\colon M \to \mathbb{C}^2$ such that (i) the preimage of the origin is a compact irreducible holomorphic curve $S \subset M$ and (ii) the map σ is one-to-one between $M \smallsetminus S$ and $\mathbb{C}^2 \smallsetminus \{0\}$. These two requirements together imply rather specific properties of M and S; cf. with Remark 8.6 below.

One such construction goes as follows. Consider the *canonical map* from $\mathbb{C}^2 \smallsetminus \{0\}$ to the projective line $\mathbb{P}^1$ that associates each point $(x,y) \neq (0,0)$ different from the origin, with the line $\{(tx,ty)\colon t \in \mathbb{C}\}$ passing through this point. The graph of this map is a complex 2-dimensional surface in the complex 3-dimensional manifold (the Cartesian product) $\mathbb{C}^2 \times \mathbb{P}^1$. The graph is not closed; to construct the closure, one has to add the *exceptional curve* $\mathbb{E} = \{0\} \times \mathbb{P}^1 \subset \mathbb{C}^2 \times \mathbb{P}^1$. The result is a nonsingular surface which we will denote by $\mathbb{M}$: by construction it is embedded in the complex 3-dimensional space $\mathbb{C}^2 \times \mathbb{P}^1$ and carries the compact curve (complex projective line, Riemann sphere) $\mathbb{E} \cong \mathbb{P}^1 \cong \mathbb{S}^2$ on it. The Cartesian projection $\mathbb{C}^2 \times \mathbb{P}^1 \to \mathbb{C}^2$ on the first component, after restriction on $\mathbb{M}$ becomes a holomorphic map

$$\sigma\colon \mathbb{M} \to \mathbb{C}^2, \qquad \sigma(\mathbb{E}) = \{0\} \in \mathbb{C}^2,$$

which is by construction one-to-one between $\mathbb{M} \smallsetminus \mathbb{E}$ and $\mathbb{C}^2 \smallsetminus \{0\}$.

Definition 8.4. The map $\sigma\colon \mathbb{M} \to \mathbb{C}^2$ between two 2-dimensional complex manifolds is called the (standard) *monoidal* map. The analytic curve $\mathbb{E} \subset \mathbb{M}$ is referred to as the (standard) *exceptional divisor*. The inverse map $\sigma^{-1}\colon \mathbb{C}^2 \smallsetminus \{0\} \to \mathbb{M} \smallsetminus \mathbb{E}$ is called the (standard) blow-up map, or simply the blow-up. A less frequently used term for the map σ is *blow-down*.

To see why $\mathbb{M}$ is a nonsingular manifold (and justify the assertions on the closure and smoothness), we produce a convenient ("standard") atlas on $\mathbb{M}$. Let z, w be two affine charts on the Riemann sphere $\mathbb{P}^1$, which take the line passing through the point $(x, y) \neq (0, 0)$ into the numbers $z = y/x$ and $w = x/y$ respectively: by construction, $w = 1/z$. These charts induce two affine charts in the respective domains V_1, V_2 on the Cartesian product $\mathbb{C}^2 \times \mathbb{P}^1$. In these charts the graph of the canonical map is given by the equations

$$y - xz = 0, \qquad \text{resp.}, \qquad x - wy = 0, \qquad (x, y) \neq (0, 0).$$

The surfaces defined by these equations, clearly remain nonsingular after extension on the line $\{x = 0,\ y = 0\} \subseteq \mathbb{C}^3$. Moreover, the functions (x, z) in the chart V_1 and (y, w) in chart V_2 respectively, become two coordinate systems (charts) on $\mathbb{M}$, defined in the two domains $U_i = \mathbb{M} \cap V_i$, $i = 1, 2$. The transition map between these charts is a monomial transformation

$$y = zx, \quad w = 1/z, \qquad \text{and reciprocally,} \qquad x = wy, \quad z = 1/w. \tag{8.2}$$

Thus $\mathbb{M}$ is indeed a nonsingular 2-dimensional complex analytic manifold. It remains to observe that the map $\sigma\colon \mathbb{M} \to \mathbb{C}^2$ in these charts is polynomial, hence globally holomorphic: $\sigma|_{U_i} = \sigma_i$, $i = 1, 2$, where

$$\sigma_1\colon (x, z) \mapsto (x, xz), \qquad \text{resp.}, \qquad \sigma_2\colon (y, w) \mapsto (yw, y). \tag{8.3}$$

The exceptional divisor $\mathbb{E}$ in the respective charts is given by the equations

$$\mathbb{E} \cap U_1 = \{x = 0\}, \qquad \text{resp.}, \qquad \mathbb{E} \cap U_2 = \{y = 0\}.$$

Remark 8.5. The formulas (8.2) and (8.3) are *real* algebraic, thus defining at the same time the real counterpart of the above construction. The real projective line $\mathbb{R}P^1$ is diffeomorphic to the circle $\mathbb{S}^1$, so the surface ${}^{\mathbb{R}}\mathbb{M}$ is constructed as a submanifold of the cylinder $\mathbb{R}^2 \times \mathbb{S}^1$. This submanifold is homeomorphic to the Möbius band. Having this analogy in mind, we will often refer to $\mathbb{M}$ as the *complex Möbius band*.

Remark 8.6. Nontriviality of the construction becomes even more striking in the complex domain. Note that the exceptional divisor cannot be *globally* defined by a single equation $\{f = 0\}$ with a function f holomorphic on $\mathbb{M}$ near $\mathbb{E}$. Indeed, if such a function exists, it would uniquely define a function

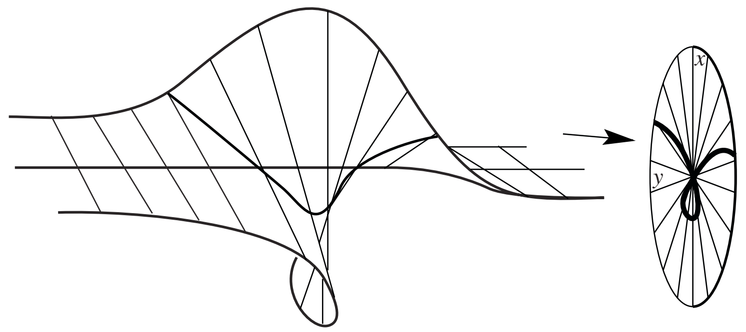


Figure I.5. Real Möbius band and its projection on $\mathbb{R}^2$ which is one-to-one outside the origin and blows down the circle $\mathbb{R}P^1 \cong \mathbb{S}^1$ into the origin

$f \circ \sigma^{-1}$ on $(\mathbb{C}^2, 0) \smallsetminus \{0\}$. This function is holomorphic and nonvanishing outside the origin and, since the point has codimension 2 in $\mathbb{C}^2$, f extends holomorphically at the origin. But the zero locus of a holomorphic function cannot have codimension 2—contradiction.

Similar arguments show that $\mathbb{E}$ is *exceptional* in the following sense: it sits rigidly inside $\mathbb{M}$ and cannot be deformed. Indeed, since $\mathbb{E}$ is compact, any deformation $\mathbb{E}'$ (a manifold uniformly close to $\mathbb{E}$) should necessarily also be compact, hence its image $\sigma(\mathbb{E}')$ should be a compact subset of $(\mathbb{C}^2, 0)$. This is impossible unless this image is a point, since σ is one-to-one outside the origin. The only remaining possibility is $\sigma(\mathbb{E}') = \{0\}$, i.e., $\mathbb{E}' = \mathbb{E}$.

Remark 8.7. These properties of the map $\sigma\colon (\mathbb{M}, S) \to (\mathbb{C}^2, 0)$ may seem to be caused by the artificial construction. However, one can prove that the construction of blow-up is natural and unique in the following sense. Consider *any* holomorphic map $\sigma'\colon (\mathbb{M}', \mathbb{E}') \to (\mathbb{C}^2, 0)$ defined in a neighborhood of a compact holomorphic curve $\mathbb{E}'$, which maps $\mathbb{E}'$ to a single point and is one-to-one on the complement $\mathbb{M}' \smallsetminus \mathbb{E}'$.

Assume that $\mathbb{E}'$ is irreducible. Then σ' is necessarily equivalent to the standard monoidal map σ: there exists a biholomorphic map $H\colon (\mathbb{M}, \mathbb{E}) \to (\mathbb{M}', \mathbb{E}')$ such that $\sigma = \sigma' \circ H$. (Without this assumption σ' can be equivalent to a *composition* of several monoidal maps.) In particular, the construction does not depend on the choice of the local coordinates (x, y) near the origin. The proof of these facts in the algebraic category can be found in [**Sha94**, Chapter IV, §3.4].

Using the local model provided by the standard monoidal transformation σ, we can construct a global map blowing up any finite set of points Σ on any two-dimensional complex manifold (surface) M.

Proposition 8.8. *Let M be a complex surface and $\Sigma \subset M$ a finite point set on it.*

Then there exists a holomorphic surface M' and a holomorphic map $\pi\colon M' \to M$ such that the preimage of any point from Σ is a Riemann sphere $\mathbb{E}_p = \pi^{-1}(p) \cong \mathbb{P}^1$ whereas π is one-to-one between $M' \smallsetminus \bigcup_{p\in\Sigma} \mathbb{E}_p$ and $M \smallsetminus \Sigma$.

Restriction of π on a small tubular neighborhood of each exceptional sphere $\mathbb{E}_p$ is equivalent to the standard monoidal map $\sigma\colon (\mathbb{M}, \mathbb{E}) \to (\mathbb{C}^2, 0)$ restricted on a neighborhood of the exceptional divisor $\mathbb{E}$.

The surface M' and the map π are unique modulo a biholomorphic isomorphism and the right equivalence respectively. As follows from Remark 8.7, the requirement that $\mathbb{E}_p$ are biholomorphically equivalent to the Riemann sphere, can be relaxed to a mere irreducibility.

The inverse map $\pi^{-1}\colon M \smallsetminus \Sigma \to M'$ is called the *simple blow-up* of the locus (finite point set) Σ. The map π itself is sometimes called a simple blow-down.

Proof of Proposition 8.8. If $M = \mathbb{C}^2$ is the standard plane, then one might try to prove the possibility of *simultaneous blow-up of several points*, constructing a suitable polynomial map by interpolation.

Yet in the category of abstract holomorphic manifolds the construction of the map π from local monoidal transformations is trivial (tautological). Consider an atlas of charts $\{U_\alpha\}$ on M including special charts U_p identifying neighborhoods of each point $p \in \Sigma$ with a neighborhood $(\mathbb{C}^2, 0)$ of the origin. Without loss of generality we can assume that all other charts do not intersect the locus Σ. The manifold M can be then described as the quotient space of the disjoint union, $M = \bigsqcup_\alpha U_\alpha / \sim$ by the equivalence relationship $\sim$ (images of the same points in different charts are identified). The manifold M' in these terms can be described as follows. Replace each special chart U_p by the neighborhood $U'_p = (\mathbb{M}, \mathbb{E})_p$, and consider again the disjoint union $\bigsqcup_\alpha U'_\alpha$ with $U'_\alpha = U_\alpha$ when the chart does not intersect Σ. The equivalence relationship $\sim$ lifts to an equivalence relationship $\sim'$ on the new disjoint union (all nonsingular points have unique preimages in U'_α). The quotient space $M' = \bigsqcup_\alpha U'_\alpha / \sim'$ by construction is a manifold. There are natural holomorphic maps $\pi\colon U'_\alpha \to U_\alpha$ which coincide with the monoidal map σ if the chart U_α was special, and identical otherwise. Clearly these maps agree with the equivalences $\sim, \sim'$ and hence define a holomorphic map $\pi\colon M' \to M$ with the required local properties. □

8C. Blow-up of analytic curves and singular foliations. As any holomorphic map, the standard monoidal map $\sigma\colon (\mathbb{M}, \mathbb{E}) \to (\mathbb{C}^2, 0)$ carries holomorphic functions and forms (by pullback) and analytic subsets (by preimages) from $(\mathbb{C}^2, 0)$ to the surface $\mathbb{M}$. However, the results are quite degenerate on the exceptional divisor $\mathbb{E}$.

The alternative is to carry the objects from the *punctured* plane $\mathbb{C}^2 \smallsetminus \{0\}$ to the *complement* $\mathbb{M} \smallsetminus \mathbb{E}$ of the exceptional divisor, and then *extend* them in one way or another on $\mathbb{E}$. The result is called the *blow-up* (*desingularization*) of the initial object.

The accurate construction is slightly different for analytic curves and for (singular) holomorphic foliations.

8**C**$_1$. *Blow-up of analytic curves.* Recall that σ^{-1} is a well-defined holomorphic map of $\mathbb{C}^2 \smallsetminus \{0\}$ to $\mathbb{M} \smallsetminus \mathbb{E}$.

Definition 8.9. The *blow-up* of an analytic curve $\gamma \subseteq (\mathbb{C}^2, 0)$ is the closure (in $\mathbb{M}$) of the preimage of the *punctured* curve $\gamma \smallsetminus \{0\}$:

$$\widetilde{\gamma} = \overline{\sigma^{-1}(\gamma \smallsetminus \{0\})}. \tag{8.4}$$

We have to verify that the result $\widetilde{\gamma}$ is an analytic curve in $\mathbb{M}$. The proof is obtained by *explicitly computing* the blow-up.

Proposition 8.10. *The blow-up of any analytic curve is again an analytic curve in* $(\mathbb{M}, \mathbb{E})$ *intersecting the exceptional divisor* $\mathbb{E}$ *only at isolated points.*

Proof. The equation of the blow-up in $\mathbb{M}$ is obtained by pulling back the equation of γ and cancelling out all terms vanishing identically on $\mathbb{E}$. However, because of the special properties of $\mathbb{E}$ in $\mathbb{M}$ (see Remark 8.6), it can be done only locally.

Consider any holomorphic germ f defining γ and its pullback $f' = \sigma^* f = f \circ \sigma \in \mathcal{O}(\mathbb{M})$. For each point $a \in \mathbb{E}$ the germ of $f' \in \mathcal{O}(\mathbb{M}, a)$ in the local ring $\mathcal{O}(\mathbb{M}, a)$ vanishes identically on $\mathbb{E}$ and can be divided by the maximal power g^ν, $\nu \geqslant 1$, where $g \in \mathcal{O}(\mathbb{M}, a)$ is any irreducible germ locally defining $\mathbb{E} = \{g = 0\}$ near a. After division we obtain the germ $\widetilde{f} = g^{-\nu} f \in \mathcal{O}(\mathbb{M}, a)$ with the following properties:

(1) *outside* $\mathbb{E}$ the germs (at a) of the loci $\sigma^{-1}(\gamma) = \{f' = 0\}$ and $\widetilde{\gamma} = \{\widetilde{f} = 0\}$ coincide,

(2) $\widetilde{f}|_{\mathbb{E}} \not\equiv 0$, hence $\mathbb{E} \not\subseteq \widetilde{\gamma}$.

If the germ $\widetilde{f} \in \mathcal{O}(\mathbb{M}, a)$ is invertible, then the germs of both $\widetilde{\gamma}$ and γ at a are both empty. If $\widetilde{f}$ is noninvertible, then $\widetilde{\gamma} = \sigma^{-1}(\gamma \smallsetminus \{0\}) \cup \{a\}$, that is, the *analytic curve* $\widetilde{\gamma}$ is a one-point closure of the preimage of $\gamma \smallsetminus \{0\}$. □

The blow-up can be alternatively described as the smallest analytic curve $\widetilde{\gamma} \subset \mathbb{M}$ such that $\sigma(\widetilde{\gamma}) = \gamma$. Note that in general this curve can be nonconnected.

8**C**$_2$. *Blow-up of foliations.* Let $\mathcal{F}$ be a singular holomorphic foliation of $(\mathbb{C}^2, 0)$ defined by a holomorphic Pfaffian form $\omega \in \Lambda^1(\mathbb{C}^2, 0)$. By definition, this means that $\mathcal{F}$ is a nonsingular holomorphic foliation of the punctured neighborhood $(\mathbb{C}^2, 0) \smallsetminus \{0\}$. Its preimage $\sigma^{-1}(\mathcal{F})$ is a nonsingular foliation of $\mathbb{M} \smallsetminus \mathbb{E}$ generated by the 1-form $\sigma^*\omega$. But since $\operatorname{codim} \mathbb{E} = 1$, by Theorem 2.20 this preimage foliation can be extended as a singular holomorphic foliation $\sigma^*\mathcal{F}$ with isolated singular points on $\mathbb{E}$.

Definition 8.11. The *blow-up of a singular foliation* $\mathcal{F}$ of $(\mathbb{C}^2, 0)$ is the singular holomorphic foliation $\widetilde{\mathcal{F}} = \sigma^*\mathcal{F}$ of $\mathbb{M}$ extending the preimage foliation $\sigma^{-1}(\mathcal{F})$ of $\mathbb{M} \smallsetminus \mathbb{E}$.

One may have two apriori possibilities for the blow-up $\widetilde{\mathcal{F}}$: either the exceptional divisor $\mathbb{E}$ is a *separatrix* of $\widetilde{\mathcal{F}}$, or different points of $\mathbb{E}$ belong to different leaves of $\widetilde{\mathcal{F}}$. In the latter case leaves of $\widetilde{\mathcal{F}}$ cross $\mathbb{E}$ transversally at almost all points, with the exception of finitely many *tangency points* and isolated singularities of $\widetilde{\mathcal{F}}$.

Definition 8.12. A singular point of a holomorphic foliation $\mathcal{F}$ on $(\mathbb{C}^2, 0)$ is called *nondicritical*, if the exceptional divisor $\mathbb{E} = \sigma^{-1}(0)$ is a separatrix of the blow-up $\sigma^*\mathcal{F}$ by the simple monoidal transformation σ.

Otherwise the singular point is called *dicritical*.

It will be shown that the "generic" singularities of a given order are nondicritical, whereas dicritical singularities correspond to certain degeneracy of the principal homogeneous terms of the vector field defining the foliation.

Remark 8.13. The previous arguments can be carried out *verbatim* for any holomorphic nonconstant map $\pi\colon (M, D) \to (\mathbb{C}^2, 0)$ squeezing a holomorphic curve $D = \pi^{-1}(0)$ (eventually, singular or reducible) into the single point at the origin and one-to-one between $M \smallsetminus D$ and $(\mathbb{C}^2, 0) \smallsetminus \{0\}$. Any holomorphic foliation $\mathcal{F}$ on $(\mathbb{C}^2, 0)$ can be pulled back as a foliation $\pi^{-1}(\mathcal{F})$ on $M \smallsetminus D$ and then extended on D everywhere except for finitely many points. The resulting singular foliation on M will be denoted by $\pi^*\mathcal{F}$ and referred to as a *desingularization*, or *blow-up* of $\mathcal{F}$ by the map π.

8D. Desingularization theorem. It turns out that singular points of *any* holomorphic foliation can be completely simplified by iterated blow-ups. The following result was first discovered by Ivar Bendixson [**Ben01**] in 1901 and improved and generalized by S. Lefschetz [**Lef56, Lef68**], A. F. Andreev [**And62, And65a, And65b**] and A. Seidenberg [**Sei68**]. A. van den Essen simplified the proof considerably in [**vdE79**]; see also [**MM80**]. In [**Dum77**] F. Dumortier obtained a generalization of this theorem for smooth rather than analytic foliations and showed that tangencies can also be eliminated. Recently O. Kleban in [**Kle95**] computed the number of iterates of simple blow-ups required to *desingularize* completely an isolated singularity of a holomorphic foliation.

Recall (see Definition 4.27) that a singularity of the foliation $\mathcal{F}$ defined by the Pfaffian equation $\omega = 0$, $\omega = f\,dx + g\,dy$ with the coefficients $f, g \in \mathcal{O}(\mathbb{C}^2, 0)$ without common factors, is *elementary*, if the linearization matrix

$A = \partial F(0,0)/\partial(x,y)$ of the dual vector field $F = -g\frac{\partial}{\partial x} + f\frac{\partial}{\partial y}$ has at least one nonzero eigenvalue.

Theorem 8.14 (I. Bendixson, A. Andreev, A. Seidenberg, S. Lefschetz, F. Dumortier)**.** *For any singularity of a holomorphic foliation $\mathcal{F}$ one can construct a holomorphic surface M with an analytic curve D on it and a holomorphic map $\pi\colon (M,D) \to (\mathbb{C}^2,0)$, one-to-one between $M \smallsetminus D$ and $(\mathbb{C}^2,0) \smallsetminus \{0\}$, such that the blow-up $\pi^*\mathcal{F}$ has only elementary singularities on D.*

More precisely, the map π resolving the singularity can be constructed as a composition of finitely many simple blow-downs.

The vanishing divisor $D = \pi^{-1}(0)$ is the union of finitely many projective lines intersecting transversally, $D = \bigcup D_j$, $D_j \cong \mathbb{P}^1$, $D_i \pitchfork D_j$.

In this section we give the *constructive* proof of this result, based on the idea of van den Essen [**vdE79, MM80**]. This idea is to introduce the *multiplicities* of isolated singularities of holomorphic foliations and monitor their decrease under blow-ups.

Detailed inspection of this algorithm yields the following estimate for the complexity of the desingularization map. It is formulated in terms of *multiplicity* of a singular point of holomorphic foliation, which will be introduced in §8**G**–§8**I**.

Theorem 8.15. *The number of simple blow-ups required to resolve an isolated singularity of multiplicity μ, does not exceed $2\mu + 1$.*

A stronger result was achieved by O. Kleban in [**Kle95**]. He proved that besides resolving all singularities into elementary, in at most $\mu + 2$ steps one can eliminate all *tangency points* between the foliation $\pi^*\mathcal{F}$ and the vanishing divisor D (Theorem 8.37).

8E. Blow-up in an affine chart: computations. Let $\omega = f\,dx + g\,dy \in \Lambda^1(\mathbb{C}^2,0)$ be a holomorphic 1-form having an isolated singularity of order n. By definition, this means that the Taylor expansions of the coefficients f, g of this form begin with homogeneous polynomials f_n, g_n of degree n and at least one of these two homogeneous polynomials does not vanish identically:

$$\operatorname{ord}_0 \omega = \min\{\operatorname{ord}_0 f, \operatorname{ord}_0 g\}.$$

Consider the pullback $\sigma^*\omega$ on the complex Möbius band $\mathbb{M}$ in the coordinates (x,z) in the chart U_1. In this chart the exceptional divisor $\mathbb{E}$ is defined by the equation $\{x = 0\}$ and the map σ takes the form $\sigma_1\colon (x,z) \mapsto (x, xz)$

and pulls back the form ω to $\omega_1 = \sigma_1^*\omega$ as follows:

$$\begin{aligned} \omega_1 &= [f(x,xz) + zg(x,xz)]\,dx + xg(x,xz)\,dz \\ &= x^{-1}[(\sigma_1^* h)\,dx + (\sigma_1^* g')\,dz], \\ h &= xf + yg, \qquad g' = x^2 g, \qquad h, g' \in \mathcal{O}(\mathbb{C}^2, 0). \end{aligned} \tag{8.5}$$

Both coefficients of the form ω_1 are divisible at least by x^n. However, the second coefficient is in fact even divisible by x^{n+1}. On the other hand, the first coefficient can "*accidentally*" also be divisible by x^{n+1}, if the homogeneous polynomial $h_{n+1} = xf_n + yg_n$ vanishes identically.

In order to extend the foliation $\widetilde{\mathcal{F}} = \sigma_1^{-1}(\mathcal{F})$ on the line $\mathbb{E} = \{x = 0\}$ in the chart U_1, we have to divide the coefficients of the form (8.5) by the *maximal possible* power of x so that the result will be not identically zero on $\mathbb{E}$. Thus we have two cases which correspond to dicritical and nondicritical singularities; cf. with Definition 8.12.

Proposition 8.16. *The singularity is nondicritical, if*

$$\operatorname{ord}_0(xf + yg) = 1 + \operatorname{ord}_0 \omega, \tag{8.6}$$

and dicritical, if

$$\operatorname{ord}_0(xf + yg) > 1 + \operatorname{ord}_0 \omega. \tag{8.7}$$

The *homogeneous* polynomial $h_{n+1} = xf_n + yg_n$ of degree $n+1$ will play an important role in computations pertinent to the dicritical case. It will be referred to as the *tangent form* for lack of a better name. The roots of h_{n+1} can be identified with the points of the projective line $\mathbb{P}$ *globally* isomorphic to the exceptional divisor $\mathbb{E}$.

Proof of the proposition. 1. In the first case (8.6) the blow-up of $\mathcal{F}$ in the chart U_1 is given by the Pfaffian equation with isolated singularities

$$\widetilde{\omega}_1 = 0, \qquad \widetilde{\omega}_1 = [h_{n+1}(1,z) + x(\cdots)]\,dx + x[g_n(1,z) + x(\cdots)]\,dz, \tag{8.8}$$

where f_n, g_n and $h_{n+1} = xf_n + yg_n$ are the homogeneous bivariate polynomials from $\mathbb{C}[x,y]$ as above and the dots denote some holomorphic functions of x and z.

The line $\mathbb{E} = \{x = 0\}$ is integral for the line field $\widetilde{\omega}_1 = 0$, so the exceptional divisor $\mathbb{E}$ in the nondicritical case is a *separatrix* of the blow-up foliation $\widetilde{\mathcal{F}}$. The singular locus $\Sigma = \operatorname{Sing}(\sigma^*\mathcal{F})$ consists of the isolated roots of the equation

$$\Sigma = \{x = 0,\ z = z_j\}, \qquad h_{n+1}(1, z_j) = 0. \tag{8.9}$$

Their number (counted with multiplicities) is equal to $\deg_z h_{n+1}(1,z)$ which can be *less* than $n+1$ if the homogeneous polynomial $h_{n+1}(x,y)$ is divisible by x. In the latter case the point $z = \infty \in \mathbb{P}^1$ is singular and should be studied in the second affine chart U_2 on $\mathbb{M}$. Globally the singular locus $\Sigma \subset \mathbb{P}^1$ is defined by the tangent form h_{n+1} as the *projective* locus in the homogeneous coordinates $\{(x : y) \in \mathbb{P}^1 \colon h_{n+1}(x, y) = 0\}$. There is a simple sufficient condition guaranteeing that a point $a \in \Sigma$ is elementary (Proposition 8.18 below).

2. In the second case (8.7) the tangent form vanishes identically, $h_{n+1} \equiv 0$, and the Pfaffian form with isolated singularities which defines the blow-up foliation in the affine chart U_1, is

$$\widetilde{\omega}_1 = 0, \qquad \widetilde{\omega}_1 = [h_{n+2}(1, z) + x(\cdots)]\, dx + [g_n(1, z) + x(\cdots)]\, dz. \quad (8.10)$$

Outside the set $T = \{g_n(1, z) = 0\} \subset \mathbb{E}$ the form $\widetilde{\omega}_1$ is nonsingular and *transversal* to $\mathbb{E}$, which means that the leaves of the blow-up foliation cross $\mathbb{E}$ transversally outside T. Note that $g_n \not\equiv 0$; otherwise the condition $h_{n+1} \equiv 0$ would mean that $f_n \equiv 0$ in violation of the assumption that the order of ω is exactly equal to n.

The points of T may correspond to either *tangency points* if $h_{n+2}(1, z)$ does not vanish (and hence the point is nonsingular), or true singularities if both $g_n(1, z)$ and $h_{n+2}(1, z)$ vanish simultaneously there. □

Remark 8.17. If the singularity is nondicritical and the tangent form $h_{n+1}(1, z)$ has degree $n + 1$ and only simple roots, the exceptional divisor $\mathbb{E}$ carries exactly $n + 1$ singular points of $\widetilde{\mathcal{F}}$. The fundamental group of the complement $\mathbb{E} \smallsetminus \Sigma$ is generated by small loops around these singularities. Hence the holonomy group of the foliation $\widetilde{\mathcal{F}}$ along the leaf $\mathbb{E} \smallsetminus \Sigma$ is generated by $n+1$ germs $g_0, \ldots, g_n \in \operatorname{Diff}(\mathbb{C}^1, 0)$ subject to a single relationship $g_0 \circ \cdots \circ g_n = \mathrm{id}$. This group is sometimes referred to as the *vanishing holonomy group* of the initial singular point of the foliation $\mathcal{F}$. Later, in §23**D**, we will discuss necessary and sufficient conditions for a group generated by $n + 1$ conformal germs to be a vanishing holonomy group of a foliation satisfying the above assumptions.

Another computation will be required in the proof of the Desingularization theorem.

Proposition 8.18. *Each* simple (*nonmultiple*) *linear factor* $ax + by$ *of the tangent form* $h_{n+1} = xf_n + yg_n$ *corresponds to an elementary singularity* $z = -a/b$ (*resp.,* $w = -b/a$) *of the blow-up foliation.*

Proof. In the assumptions of the proposition, the singularity is obviously nondicritical and without loss of generality we may assume that the factor is simply y, and $h_{n+1}(1, z) = zu(z)$ and $u(0) = 1$.

The vector spanning the same line field as the distribution (8.8), has the form

$$\dot{z} = z + ax + \mathfrak{m}^2, \qquad \dot{x} = -bx + \mathfrak{m}^2,$$

where a, b are some two complex numbers and $\mathfrak{m}^2$ denote functions of order $\geqslant 2$. The linearization matrix $\left(\begin{smallmatrix} 1 & * \\ 0 & * \end{smallmatrix}\right)$ of this field has nonzero eigenvalue 1 for the eigenvector tangent to $\mathbb{E}$. □

8F. Divisors. To proceed with the demonstration of the desingularization theorem, we first introduce a convenient algebraic formalism for counting analytic subvarieties (points and analytic hypersurfaces) with certain integer *multiplicities* (positive or negative). While this formalism cannot be easily extended for subvarieties of intermediate dimensions, for the two extreme dimensions (zero and maximal, i.e., codimension 1) the theory is as complete as possible.

The integer multiplicity can be easily attached to analytic subvarieties of codimension one (hypersurfaces) using the fact that the ring of germs of analytic functions admits unique irreducible factorization. This construction leads to the notion of a *divisor*, introduced and discussed in this section. Multiplicity of zero-dimensional sets (isolated points) can be introduced in a different way via codimension of the respective ideals as explained in §8**G** as the *intersection multiplicity* of two analytic curves. Behavior of these multiplicities under blow-up is studied in §8**H**–§8**I**.

8$\mathbf{F}_1$. *Definitions.* A *divisor* on a complex manifold M is a finite union of irreducible analytic hypersurfaces (analytic subsets of codimension 1) with assigned integer multiplicities (coefficients). By this definition, each divisor D is a formal sum $\sum_\gamma k_\gamma\, \gamma$ where the summation is formally over *all* irreducible subvarieties of codimension 1, but only finitely many integer coefficients $k_\gamma \in \mathbb{Z}$ can be in fact nonzero. Divisors form an Abelian group denoted by $\operatorname{Div}(M)$ with the operation denoted additively, $\left(\sum k_\gamma\, \gamma\right) + \left(\sum k'_\gamma\, \gamma\right) = \sum (k_\gamma + k'_\gamma)\, \gamma$. The divisor is called *effective* if all k_γ are nonnegative; any divisor can be formally represented as a formal *difference* of two effective divisors. The *support* of a divisor is the union of all subvarieties entering into D with nonzero coefficients,

$$|D| = \bigcup_{k_\gamma \neq 0} \gamma \cong \sum_{k_\gamma \neq 0} \gamma,$$

which can be alternatively thought of as either the point set or an effective divisor with all k_γ being just 0 or 1.

If M is one-dimensional, divisors are finite point sets with integer multiplicities attached to each point. We will be interested here in the two-dimensional case where M is a holomorphic surface and the divisors are unions of irreducible curves counted with multiplicities.

8$\mathbf{F}_2$. *Divisors and meromorphic functions.* Each holomorphic function $f \in \mathcal{O}(M)$ defines an effective divisor D_f called the *divisor of zeros* of f as follows. The support $|D_f|$ is the zero locus $Z_f = \{f = 0\} \subseteq M$, and if the germ of f at a point $a \in M$ has the irreducible factorization $f = \prod f_j^{\nu_j}$ in the local ring $\mathcal{O}(M, a)$, then the component $D_j = D_{f_j}$ of D_f is assigned the multiplicity $\nu_j \geqslant 0$:

$$D_f = \sum_j \nu_j D_j, \qquad D_j = D_{f_j} = \{f_j = 0\}.$$

This definition allows us to assign the multiplicity ν_j to each irreducible component $D_j \subseteq D_f$ near the point a only, but the answer is obviously locally constant as a varies along D_j. Since D_j is connected, the result does not depend on a, moreover, one can always choose a being a smooth point on D_j.

For a meromorphic function $h = f/g$ the divisor D_h is defined as the formal difference,

$$D_{f/g} = D_f - D_g.$$

It obviously does not depend on the choice of the representation.

Conversely, any divisor can be associated with a meromorphic function, albeit only locally. Let $D = \sum k_\gamma \gamma$ be a divisor on M. Then M can be covered by a union of open domains $\{U_\alpha\}$ so that in each domain U_α each hypersurface $\gamma \subseteq |D|$ is represented by a holomorphic equation $\{f_{\alpha,\gamma} = 0\}$ with the differential $df_{\alpha,\gamma}$ nonvanishing outside a set of codimension 2 on γ. The divisor D locally in U_α is defined by the meromorphic function $f_\alpha = \prod_\gamma f_{\alpha,\gamma}^{k_\gamma} \in \mathcal{M}(U_\alpha)$. The collection $\{f_\alpha\}$ is called a *meromorphic* 1-*cochain* defining the divisor D.

Consider the pairwise intersections $U_{\alpha\beta} = U_\alpha \cap U_\beta$ and the ratios $g_{\alpha\beta} = f_\alpha / f_\beta$ in these intersections. These ratios are holomorphic and nonvanishing, since both f_α and f_β define the same divisor in the intersection $U_{\alpha\beta}$. The collection of holomorphic invertible functions $g_{\alpha\beta}$ is called the *holomorphic cochain* (more precisely, holomorphic 2-cochain) defining the divisor D. Addition of divisors corresponds to multiplication of the holomorphic cochains: if D, D' are two divisors defined by the holomorphic cochains $\{g_{\alpha\beta}\}$ and $\{g'_{\alpha\beta}\}$, then the sum $D + D'$ is defined by the cochain $\{g_{\alpha\beta} g'_{\alpha\beta}\}$.

Note that some divisors *may not be definable* by a single *global* equation on M, e.g., the exceptional divisor $\mathbb{E}$ on the complex Möbius band $\mathbb{M}$; see Remark 8.6.

With respect to holomorphic maps, divisors behave like analytic functions, i.e., they are *pulled back* by such maps. Let $\pi \colon M' \to M$ be a nonconstant holomorphic map between two connected manifolds of the same dimension and $D = \sum k_\gamma \gamma$ a divisor on M defined by the meromorphic cochain $\{f_\alpha\}$.

Definition 8.19. The *preimage* (pullback) $\pi^{-1}(D)$ of a divisor $D \in \operatorname{Div}(M)$ is the divisor on M' which in the open domains (charts) $U'_\alpha = \pi^{-1}(U)$ is defined by the meromorphic cochain $f'_\alpha = \pi^* f_\alpha \in \mathbf{M}(U'_\alpha)$.

Since π^* is a ring homomorphism, taking preimages commutes with addition/subtraction of divisors: for any two divisors D, D' on M,

$$\pi^{-1}(D \pm D') = \pi^{-1}(D) \pm \pi^{-1}(D').$$

In other words, $\pi^{-1} \colon \operatorname{Div}(M) \to \operatorname{Div}(M')$ is a homomorphism of Abelian groups.

Example 8.20. Preimage of the sum of n different straight lines $\sum_1^n \ell_j$ associated with the function $f(x,y) = \prod l_j \in \mathcal{O}(\mathbb{C}^2, 0)$ (the product of n different linear factors) by the monoidal map $\sigma \colon \mathbb{M} \to \mathbb{C}^2$ is the divisor $n\mathbb{E} + \sum_1^n \widetilde{\ell}_j$, where $\mathbb{E}$ is the exceptional divisor and $\widetilde{\ell}_j$ the blow-ups of the lines ℓ_j.

8G. Intersection multiplicity and intersection index. In this section we define the multiplicity of intersection of two divisors (curves) at an isolated point and the global intersection index between divisors. More details can be found in [**vdE79, MM80, Chi89**]. The theorem on equivalence of different definitions appears in [**AGV85**, §5], and the intersection theory in the algebraic context is explained in [**Sha94**, Chapter IV].

We start with the particular case of effective divisors and define first the local multiplicity of their intersection at a common point, say, the origin in $\mathbb{C}^2$. Let $f, g \in \mathcal{O}(\mathbb{C}^2, 0)$ be two holomorphic germs and D_f, D_g the respective *effective* divisors in $(\mathbb{C}^2, 0)$. We say that the intersection D_f and D_g is *isolated* at the origin, if $|D_f| \cap |D_g| \cap (\mathbb{C}^2, 0) = \{0\}$ (in the sense of germs of analytic sets). The intersection is isolated if and only if no irreducible component enters both divisors with positive coefficient, i.e., f, g have no common irreducible factors in the ring of germs $\mathcal{O}(\mathbb{C}^2, 0)$. In this case we can give several equivalent definitions of the intersection multiplicity $\mu = D_f \mathbin{\S} D_g$ between D_f and D_g at the origin $a = 0$.

8**G**$_1$. *Algebraic construction.* Consider the ideal $I_{f,g} = \langle f, g \rangle \subset \mathcal{O}(\mathbb{C}^2, 0)$ generated by these germs in the local ring of germs, and the quotient *local algebra* $Q_{f,g} = \mathcal{O}(\mathbb{C}^2, 0)/I_{f,g}$ as a linear space over $\mathbb{C}$. The *algebraic multiplicity of intersection* is defined as the dimension of the local algebra (codimension of the ideal),

$$\begin{aligned} D_f \mathbin{\S} D_g &= \dim_{\mathbb{C}} Q_{f,g} = \operatorname{codim}_{\mathcal{O}(\mathbb{C}^2,0)} I_{f,g}, \\ I_{f,g} = \langle f, g \rangle \subset \mathcal{O}(\mathbb{C}^2, 0), &\qquad Q_{f,g} = \mathcal{O}(\mathbb{C}^2, 0)/I_{f,g}. \end{aligned} \tag{8.11}$$

By definition, the equality $\dim Q_{f,g} = \mu < +\infty$ means that there exist the germs $e_1, \dots, e_\mu$ which are a basis of the local algebra so that any other germ $u \in \mathcal{O}(\mathbb{C}^2, 0)$ admits the representation

$$u = \sum_{1}^{\mu} c_i\, e_i + af + bg, \qquad c_1, \dots, c_\mu \in \mathbb{C}, \quad a, b \in \mathcal{O}(\mathbb{C}^2, 0), \tag{8.12}$$

and the constant coefficients c_i are defined uniquely. By this definition, the multiplicity of intersection depends only on the ideal $\langle f, g \rangle$.

8$\mathbf{G}_2$. *Geometric construction.* The pair of analytic functions (f, g) considered as coordinate functions, defines a holomorphic map $P = P_{f,g} \colon (\mathbb{C}^2, 0) \to (\mathbb{C}^2, 0)$. If the intersection D_f and D_g is isolated, the preimage of the $P^{-1}(0,0) = (0,0)$ is a single point. Maps with such properties have an integer topological invariant, the degree. Consider a small 3-dimensional real sphere $\mathbb{S}^3_\rho = \{|x|^2 + |y|^2 = \rho\} \subset \mathbb{C}^2 \cong \mathbb{R}^4$ and the "normalization" of P, the map $\widehat{P} = \widehat{P}_{f,g} \colon (x, y) \mapsto P(x,y)/|P(x,y)|$. The normalized map $\widehat{P}$ is not analytic, only differentiable, and its range is the unit sphere $\mathbb{S}^3_1 = \{|z|^2 + |w|^2 = 1\}$. Restricting $\widehat{P}$ on a sufficiently small sphere $\mathbb{S}^3_\rho$, we obtain thus a map between two spheres has an invariant, the *topological degree*, which can be described as the number of preimages (counted with the sign determined by the orientation) of a generic point in the target sphere. This degree is the same for all sufficiently small choices of $\rho > 0$.

The *geometric multiplicity* of intersection between D_f and D_g at the origin is defined as the topological degree of the map $\widehat{P}$,

$$\begin{gathered} D_f \mathbin{\text{⁝}} D_g = \operatorname{top\,deg}_0 \widehat{P}_{f,g}, \qquad \widehat{P}_{f,g} \colon \mathbb{S}^3_\rho \to \mathbb{S}^3_1, \\ \widehat{P}_{f,g} \colon (x, y) \mapsto \frac{\big(f(x,y), g(x,y)\big)}{|f(x,y)|^2 + |g(x,y)|^2}. \end{gathered} \tag{8.13}$$

8$\mathbf{G}_3$. *Deformational construction.* Let the positive number $\rho > 0$ be so small that the system of equations $\{f = 0,\ g = 0\}$ has a unique solution $\{x = y = 0\}$ in the ball $B_\rho = \{|x|^2 + |y|^2 < \rho\}$ (as before, $f, g \in \mathcal{A}(B_\rho)$ are holomorphic representatives of the initial germs). Then for almost all sufficiently small (relative to ρ) complex values $a, b \in \mathbb{C}$, $|a|, |b| < \varepsilon$, the holomorphic level curves $\{f = a\}$ and $\{g = b\}$ are smooth inside B_ρ and intersect transversally. This follows from the Sard lemma: it is sufficient to require a be a regular value for f and b a regular value of g restricted on the nonsingular curve $\{f = a\}$. The transversality implies that the intersection $\{f = a\} \cap \{g = b\} \cap B_\rho$ consists of isolated points. The *deformational multiplicity of intersection* between D_f and D_g at the origin is the number of these points:

$$D_a \mathbin{\text{⁝}} D_g = \#\{f = a\} \cap \{g = b\} \cap B_\rho \quad \text{for generic } (a, b) \in (\mathbb{C}^2, 0). \tag{8.14}$$

Apriori it is not clear why this definition makes sense and the above number is the same for *all* generic combinations (a, b).

8$\mathbf{G}_4$. *Definition of multiplicity.* One of the central results of the singularity theory claims that the three definitions of multiplicity lead to the same answer.

Theorem 8.21. *For a pair of germs $f, g \in \mathcal{O}(\mathbb{C}^2, 0)$ without common factors in the irreducible decomposition, all three definitions* (8.11), (8.13) *and* (8.14) *lead to the same* finite *number $\mu = \mu_{f,g} \in \mathbb{Z}_+$.* □

The proof of this theorem can be found in [**AGV85**, §5].

Definition 8.22. The common value established in Theorem 8.21 is called the multiplicity of intersection between the divisors D_f and D_g at the origin.

Remark 8.23. The ideas behind the proof of Theorem 8.21 are rather natural and can be explained as follows.

Coincidence between the geometric and deformational definitions is actually the theorem about the sum of indices of singular points of a vector field $P_{f-a,g-b} \in \mathcal{D}(B_\rho)$ with the coordinates $(f - a, g - b)$ in the ball B_ρ that is equal to the degree of this vector field on the boundary of the ball. Important is the fact that each transversal intersection in the complex domain corresponds to a singular point of index $+1$ (unlike the real case where the index can be of positive and negative sign). The degree of the vector field $P_{f-a,g-b}$ on the boundary is an integer-valued function of a, b that is continuous, hence it must be a constant equal to the limit, the degree of $P_{f-0,g-0}$ which is the geometric multiplicity (8.13). This argument can be made into a rigorous proof that the geometric and deformational multiplicities coincide.

If in the definition of the algebraic multiplicity we replace the germs f, g by the holomorphic functions $f-a$ and $g-b$ considered as elements from the ring $\mathcal{A}(B_\rho)$ for some positive ρ, then the quotient algebra $\mathcal{A}(B_\rho)/\langle f - a, g - b\rangle$ is isomorphic to the algebra of functions on μ distinct points, where μ is the deformational multiplicity given by (8.14). It requires some effort to prove that the dimension of the quotient algebra remains the same, first in the limit as $(a, b) \to 0 \in \mathbb{C}^2$, and then in the limit $\rho \to 0^+$. The latter is exactly the algebraic multiplicity.

A convenient tool for computation of the intersection multiplicity is the following lemma. Assume that the divisor D_f is irreducible (i.e., the germ f is irreducible in the local ring $\mathcal{O}(\mathbb{C}^2, 0)$. In this case D_f can be locally parameterized by an injective nonconstant map $\tau\colon (\mathbb{C}^1, 0) \to (\mathbb{C}^2, 0)$ such that $0 \equiv f \circ \tau \in \mathcal{O}(\mathbb{C}^1, 0)$ (see Theorem 2.26).

Lemma 8.24. *The intersection of an irreducible local divisor D_f with another effective local divisor D_g is isolated if and only if the germ $g \circ \tau$ is not identically zero, and the multiplicity $D_f \mathbin{⚲} D_g$ of this intersection is equal to the order* $\operatorname{ord}_0(f \circ \tau)$.

Proof. Consider a regular value b of the function g on the curve $\gamma = \{f = 0\}$ and the corresponding intersection locus $Z_{0b} = \{f = 0,\ g = b\}$ inside B_ρ. We prove first that the intersection multiplicity $\mu = D_f \mathbin{⚲} D_g$ is equal to the number $\#Z_{0b}$ of the points in this locus.

To that intermediate end, consider the coefficient $h \in \mathcal{A}(B_\rho)$ of the 2-form $df \wedge dg = h\, dx \wedge dy$. This coefficient cannot vanish identically on γ:

by irreducibility of f, the differential $df|_\gamma$ vanishes only at the origin, hence $h \equiv 0$ would mean that dg is proportional to df at all points of γ, therefore $dg|_\gamma \equiv 0$ and $g|_\gamma$ is a constant. As $g(0) = 0$, this constant is necessarily equal to zero, in contradiction with our assumptions that Z_{00} consists of a single point at the origin. Thus $h|_\gamma \not\equiv 0$, and one can assume without loss of generality that ρ is so small that $h|_\gamma$ is nonvanishing outside the origin.

Nonvanishing of h at all points $Z_{0b} \subseteq \gamma$ for $b \neq 0$ means that the restriction of f on the curve $\{g = b\}$ has simple roots at exactly these points. Any small perturbation $f - a$ will have exactly the same number $\#Z_{ab} = \#Z_{0b}$ of complex roots on $\{g = b\}$ which is by deformational definition of multiplicity equal to μ.

The points from Z_{0b} are τ-parameterized by the small roots of the holomorphic function of one variable $(g - b) \circ \tau = g \circ \tau - b$ which is a small perturbation of the function $g \circ \tau$. It remains to observe that a small perturbation of a germ of order μ in $\mathcal{O}(\mathbb{C}^1, 0)$ is a function that has exactly μ roots in a sufficiently small neighborhood of the origin. □

Another application of Theorem 8.21 is the following additivity of the intersection multiplicity.

Proposition 8.25. *For any three* effective *divisors D, D', D'' on $(\mathbb{C}^2, 0)$, such that $D \cap (|D'| \cup |D''|)$ is a single point 0, the intersection multiplicities satisfy the equality*

$$D \mathbin{⁝} (D' + D'') = D \mathbin{⁝} D' + D \mathbin{⁝} D''. \tag{8.15}$$

Proof. Let D', D'' and D be the divisors of the germs f, g and h respectively, which are identified with their representatives holomorphic in a sufficiently small ball B_ρ. Then the divisor $D' + D''$ is that of the product fg.

By the deformational construction, for a generic combination of the values $(a', a'', b) \in (\mathbb{C}^3, 0)$, the intersections $Z'_{a'b} = \{f = a',\ h = b\}$ and $Z''_{a''b} = \{g = a'',\ h = b\}$ are transversal and consist of $\mu' = D \mathbin{⁝} D'$ and $\mu'' = D \mathbin{⁝} D''$ points respectively. Excluding only finitely many values of b, one may assume without loss of generality that $Z'_{a'b}$ and $Z''_{a''b}$ are disjoint: this happens if the level curve $\{h = b\}$ avoids the common points of $\{f = a'\}$ and $\{g = a''\}$. In these assumptions, the number of transversal intersections between the curve $\{h = b\}$ and the reducible curve $\{(f - a')(g - a'') = 0\}$ is exactly equal to $\mu' + \mu''$.

The function $(f - a')(g - a'')$ is not a perturbation of the form $fg - a$ that appears in the deformational construction. Yet because of the continuity, the degree of the vector fields $P_{fg-a,h-b}$, $P_{(f-a')(g-a''),h-b}$ and $P_{fg,h}$ on the boundary of the ball B_ρ are the same if a, a', a'' and b are all sufficiently small compared to ρ. Thus by the geometric definition of the multiplicity, we conclude that $D \overset{0}{:} (D' + D'') = \mu' + \mu''$. □

8$\mathbf{G}_5$. *Intersection form between arbitrary global divisors.* Using Proposition 8.25, one can extend the formulas for the multiplicity of intersections for arbitrary (not necessarily effective) divisors, by the standard construction.

For a pair of local divisors, an effective divisor D' and an arbitrary divisor D represented as a difference of two effective divisors $D = D_1 - D_2$, we define the multiplicity of intersection (always at the origin) as

$$D' \cdot D = D' \cdot D_1 - D' \cdot D_2. \tag{8.16}$$

If $D = D_3 - D_4$ is another representation, then by definition $D_1 + D_4 = D_2 + D_3$, so that by Proposition 8.25, $D' \cdot D_1 + D' \cdot D_4 = D' \cdot D_2 + D' \cdot D_3$ and hence $D_3 \cdot D' - D_4 \cdot D'$ coincides with $D' \cdot D_1 - D' \cdot D_2$, which means that the definition is self-consistent. Multiplicity of intersection of two noneffective divisors is defined by iterating this construction twice, and the additivity law (8.15) holds automatically for any three divisors.

Consider now the general case of divisors on an arbitrary *complex analytic* surface M. Two divisors D, D' on M are said to have *isolated intersection*, if $|D| \cap |D'|$ is a finite point set.

Definition 8.26. The *intersection index* between two divisors D, D' with isolated intersection is the sum of all intersection multiplicities:

$$D \cdot D' = \sum_{a \in M} D \overset{a}{:} D', \qquad \text{if } |D| \cap |D'| \text{ is a finite set.} \tag{8.17}$$

The summation in (8.17) is formally extended over all points in M, but only points from $|D| \cap |D'|$ may contribute nonzero terms.

The intersection index is a bilinear (over $\mathbb{Z}$) symmetric form $\operatorname{Div}(M) \times \operatorname{Div}(M) \to \mathbb{Z}$, also called *intersection index*, defined on pairs of divisors with isolated intersection,

$$\begin{gathered} D, D' \longmapsto D \cdot D', \qquad \text{when } |D| \cap |D'| \text{ is finite set,} \\ D \cdot (D' \pm D'') = D \cdot D' \pm D \cdot D'', \qquad (D, D') = (D', D). \end{gathered} \tag{8.18}$$

Defined in this way, the intersection index generalizes the notion of the number of intersection points counted with multiplicities. Its functoriality (behavior by holomorphic maps) is studied in the next subsection.

8H. Blow-up and intersection index. The intersection index is well defined and invariant by *biholomorphisms*: if $\pi\colon M' \to M$ is a biholomorphism, then

$$\begin{gathered} \pi^{-1}(D) \cdot \pi^{-1}(D') = D \cdot D', \\ D, D' \in \operatorname{Div}(M), \quad \pi^{-1}(D), \pi^{-1}(D') \in \operatorname{Div}(M') \end{gathered} \tag{8.19}$$

for any two divisors D, D' on M with an isolated intersection. However, if σ is a *blow-up* then the preimage of the point $\{0\}$ is the exceptional divisor which therefore belongs to the preimage of *any* divisor. Hence $\sigma^{-1}(D)$ and $\sigma^{-1}(D')$ necessarily have nonisolated intersection even if $|D| \cap |D'| = \{0\}$: this intersection always contains the exceptional divisor $\mathbb{E}$ with a positive multiplicity if D, D' were effective; see Example 8.20.

One can attempt to *extend* the intersection form on pairs of divisors $R, R' \in \operatorname{Div}(C)$ which have no *nonexceptional* common components, i.e., when

$$|R| \cap |R'| \subseteq S, \tag{8.20}$$

so that the identity (8.19) would hold also when π is a blow-up. We shall see that only one such extension is possible.

Remark 8.27. Theorem 8.21 can be interpreted as the *local continuity* of the intersection index. For instance, consider an effective divisor D defined by a family of local equations $\{f_\alpha = 0\}$ in suitable charts U_α. If another family $\{f'_\alpha \in \mathcal{O}(U_\alpha)\}$ is a sufficiently small perturbation of $\{f_\alpha \in \mathcal{O}(U_\alpha)\}$ also has nonvanishing holomorphic ratios $f'_\alpha/f'_\beta \in \mathcal{O}(U_\alpha \cap U_\beta)$, it defines a small perturbation D' of the divisor D as explained in §8$\mathbf{F}_2$. By the deformational construction, the intersection index of D and D' with any other divisor D'' is the same, $D \cdot D'' = D' \cdot D''$ (while multiplicities of particular intersection points may of course change).

Thus in principle one might wish to define the *self-intersection index* for any divisor D by perturbing it slightly to become a divisor D_ε and let by definition $D \cdot D = \lim_{D_\varepsilon \to D} D \cdot D_\varepsilon$. For instance, if D is defined by a *global* equation $D = D_f$ for some $f\colon M \to \mathbb{C}$, then one can choose $D_\varepsilon = D_{f-\varepsilon}$: since different level curves are disjoint, $D \cdot D_e = 0$ for all $\varepsilon \neq 0$, and hence we have the identity $D \cdot D = 0$. On the other hand, if $M = \mathbb{P}^2$ is the projective plane and D is a line on it, then $D \cdot D = 1$.

Yet the self-intersection index of the exceptional divisor $\mathbb{E}$ cannot be obtained in this way because of the rigidity of $\mathbb{E}$ inside the Möbius band $\mathbb{M}$ (Remark 8.6). Moreover, we will see that in order to preserve (8.19), one has to assign the self-intersection index $\mathbb{E} \cdot \mathbb{E}$ the *negative* value -1 (note that the intersection index between any two *different* divisors is always nonnegative).

Example 8.28. Consider two divisors defined by two lines $\ell_{1,2}$ transversally crossing at the origin in $(\mathbb{C}^2, 0)$. Their preimages by the standard monoidal map $\sigma\colon \mathbb{M} \to (\mathbb{C}^2, 0)$ consist of the blow-ups $\widetilde{\ell}_{1,2}$ and the exceptional divisor:

$$\sigma^{-1}(\ell_j) = \mathbb{E} + \widetilde{\ell}_j, \qquad j = 1, 2;$$

cf. with Example 8.20. Note that both blow-ups $\widetilde{\ell}_{1,2}$ are smooth, intersect $\mathbb{E}$ transversally, hence $\widetilde{\ell}_j \cdot \mathbb{E} = 1$, and are *disjoint*, so $\widetilde{\ell}_1 \cdot \widetilde{\ell}_2 = 0$. If the preimages were to have the same intersection index $\sigma^{-1}(\ell_1)\cdot\sigma^{-1}(\ell_2)=\ell_1\cdot\ell_2=1$, then we would have the identity

$$1=\ell_1\cdot\ell_2=\mathbb{E}\cdot\mathbb{E}+\mathbb{E}\cdot(\widetilde{\ell}_1+\widetilde{\ell}_2)+\widetilde{\ell}_1\cdot\widetilde{\ell}_2=\mathbb{E}\cdot\mathbb{E}+1+1+0,$$

which leaves only one possibility, $\mathbb{E}\cdot\mathbb{E}=-1$.

Theorem 8.29. *The intersection form between divisors on* $\mathbb{M}$ *can be uniquely extended for pairs of divisors satisfying* (8.20) *as a symmetric bilinear form with the following properties:*

$$\mathbb{E}\cdot\mathbb{E}=-1, \tag{8.21}$$

$$\sigma^{-1}(D)\cdot\mathbb{E}=\quad 0, \qquad \forall D\in\operatorname{Div}(\mathbb{C}^2,0), \tag{8.22}$$

$$\sigma^{-1}(D)\cdot\sigma^{-1}(D')=D\cdot D', \qquad \forall D,D'\in\operatorname{Div}(\mathbb{C}^2,0) \tag{8.23}$$

(*the last condition holds only for pairs of divisors* $D,D'\in\operatorname{Div}(\mathbb{C}^2,0)$ *having isolated intersection*).

Proof. We need to prove that the rule (8.21) if adopted as an axiom and combined with bilinearity, would imply the identities (8.22) and (8.23) for arbitrary divisors $D,D'\in\operatorname{Div}(\mathbb{C}^2,0)$. Because of the bilinearity and symmetry, it is sufficient to complete the proof when the divisor $D=D_f$ is a curve defined by a holomorphic germ $f\in\mathcal{O}(\mathbb{C}^2,0)$.

Denote by $n=\operatorname{ord}_0 f$ the order of the holomorphic germ $f=f_n+f_{n+1}+\cdots$. Without loss of generality we may assume that the principal homogeneous part f_n is *not* divisible by x, so that $f_n(x,y)=cy^n+\cdots$, $c\neq 0$ (otherwise an affine change of coordinates should first be made). In the chart U_1 we have

$$\begin{aligned}\sigma_1^*f(x,z)&=x^nf_n(1,z)+x^{n+1}(1,z)+\cdots=x^n[f_n(1,z)+xf_{n+1}+\cdots]\\&=x^n\widetilde{f}(x,z), \qquad \widetilde{f}(0,z)=f_n(1,z)\not\equiv 0,\end{aligned}$$

so that by definition of the preimage of divisors,

$$\sigma^{-1}(D_f)=n\mathbb{E}+\widetilde{D}_f, \qquad \widetilde{D}_f=D_{\widetilde{f}}, \quad n=\operatorname{ord}_0 f. \tag{8.24}$$

As a curve, $|\widetilde{D}_f|$ is the blow-up of the curve $|D_f|$, since the function $\widetilde{f}$ does not vanish identically on $\mathbb{E}$. Occurrence of the term $n\mathbb{E}$ stresses the difference between preimage of the divisor and blow-up of its support curve.

The intersection between $\widetilde{D}_f$ and $\mathbb{E}$ is isolated and consists of the roots of the polynomial $f_n(1,z)$ of degree exactly n. If $a=(0,a')$ is such a point, then the multiplicity of intersection $\widetilde{D}_f\cdot\mathbb{E}$ at this point is equal to the multiplicity of the root of $f_n(1,z)$ at $z=a'\in\mathbb{C}$, since $\widetilde{f}(x,z)=f_n(1,z) \bmod \langle x\rangle$ and the quotient rings $\mathcal{O}(\mathbb{C}^2,a)/\langle x,\widetilde{f}\rangle$ and $\mathcal{O}(\mathbb{C}^1,a')/\langle f_n(1,\cdot)\rangle$ are naturally isomorphic. Adding the contributions of all points together, we obtain

$$\widetilde{D}_f\cdot\mathbb{E}=\deg_z f_n(1,z)=\operatorname{ord} f=n. \tag{8.25}$$

Using the axiom (8.21), we obtain from (8.24) by linearity

$$\sigma^{-1}(D_f)\cdot\mathbb{E}=(-1)\cdot n+\widetilde{D}_f\cdot\mathbb{E}=-n+n=0.$$

The proof of (8.22) is complete.

To prove (8.23) we assume that the analytic curve $D=D_f$ is irreducible and parameterized by an injective holomorphic map $\tau\colon(\mathbb{C}^1,0)\to(\mathbb{C}^2,0)$, $t\mapsto\big(x(t),y(t)\big)$.

By Lemma 8.24, the intersection multiplicity $D_f \mathbin{⁝} D_g$ is equal to the multiplicity (order) $\operatorname{ord}_0 g\circ\tau$ of the root $t=0$ of the composition $g\circ\tau$.

If $D_f=\gamma$ is an irreducible curve parameterized by τ, then the map $\widetilde{\tau}\colon t\mapsto\sigma^{-1}\circ\tau$, $t\neq0$, parameterizes the points of $\sigma^{-1}(\gamma)\smallsetminus\mathbb{E}$. It obviously extends holomorphically at the origin and becomes a map $\widetilde{\tau}\colon(\mathbb{C}^1,0)\to C$ parameterizing the blow-up curve $\widetilde{D}_f=\widetilde{\gamma}$.

If $D'=D_g$ is an arbitrary divisor (reducible or not), then using Lemma 8.24 twice we obtain

$$\begin{aligned}D_g\cdot D_f&=\operatorname{ord}_0 g\circ\tau=\operatorname{ord}_0 g\circ\sigma\circ\sigma^{-1}\circ\tau=\operatorname{ord}_0(\sigma^*g)\circ\widetilde{\tau}\\&=D_{\sigma^*g}\cdot\widetilde{D}_f=\sigma^{-1}(D_g)\cdot\widetilde{D}_f.\end{aligned}$$

Combining this with (8.24) and (8.22), we obtain

$$\begin{aligned}\sigma^{-1}(D_g)\cdot\sigma^{-1}(D_f)&=\sigma^{-1}(D_g)\cdot(n\mathbb{E}+\widetilde{D}_f)\\&=n\,\sigma^{-1}(D_g)\cdot\mathbb{E}+\sigma^{-1}(D_g)\cdot\widetilde{D}_f\\&=0+D_g\cdot D_f=D_g\cdot D_f.\end{aligned}$$

The proof of (8.23) is complete when D is irreducible. As was already mentioned, the proof in the general case follows from bilinearity of the intersection index. □

As a corollary to Theorem 8.29, we obtain a simple formula for the intersection index between *blow-ups* of two analytic curves.

Corollary 8.30. *For any pair of two holomorphic curves $\gamma,\gamma'\subseteq(\mathbb{C}^2,0)$ of orders m and m', and their blow-ups $\widetilde{\gamma},\widetilde{\gamma}'\subset(\mathbb{M},\mathbb{E})$, the intersection indices are related by the formula*

$$\gamma\cdot\gamma'=\widetilde{\gamma}\cdot\widetilde{\gamma}'+mm'. \tag{8.26}$$

Proof. By (8.24), on the level of divisors

$$\sigma^{-1}(\gamma)=m\mathbb{E}+\widetilde{\gamma},\qquad\sigma^{-1}(\gamma')=m'\mathbb{E}+\widetilde{\gamma}'.$$

Using bilinearity, we conclude that

$$\widetilde{\gamma}\cdot\widetilde{\gamma}'=(\sigma^{-1}(\gamma)-m\mathbb{E})\cdot(\sigma^{-1}(\gamma')-m'\mathbb{E})=\gamma\cdot\gamma'-0m-0m'+(-1)\,mm'$$

by virtue of the three rules (8.21), (8.22) and (8.23). □

Example 8.31. If γ, γ' are two *smooth* (of order 1) analytic curves, then their intersection multiplicity decreases by 1 after blow-up. Since in the smooth case the intersection multiplicity is equal to the order of tangency between γ and γ' minus 1, *the order of tangency between smooth curves is also decreased by one by blow-up.*

8I. Blow-up and multiplicity of singular foliations. Consider a singular holomorphic foliation $\mathcal{F}$ defined by the Pfaffian equation $\{\omega = 0\}$, $\omega \in \Lambda^1(\mathbb{C}^2, 0)$ or a holomorphic vector field $F \in \mathcal{D}(\mathbb{C}^2, 0)$ near an isolated point at the origin. Denote by n the *order* of the form ω at the origin: by definition, it means that

$$\omega = f\,dx + g\,dy = (f_n + f_{n+1} + \cdots)\,dx + (g_n + g_{n+1} + \cdots)\,dy \qquad (8.27)$$

and the homogeneous polynomials f_n, g_n of lowest degree n do not vanish identically: $f_n\,dx + g_n\,dy \neq 0$. The assumption that the singularity is isolated means that the intersection of the coordinate divisors D_f and D_g is isolated.

Definition 8.32. The *multiplicity* $\mu_0(\omega)$ of the singular point of the form (8.27) at the origin is the intersection multiplicity $D_f \,\text{⸮}\, D_g$ between the respective divisors.

The multiplicity $\mu_a(\mathcal{F})$ of a singular foliation $\mathcal{F}$ at a point a is the multiplicity of any holomorphic form ω tangent to $\mathcal{F}$ and having an isolated singular point at a.

Consider a small perturbation F_ε of the vector field, say, $(f - \varepsilon_1)\frac{\partial}{\partial x} + (g - \varepsilon_2)\frac{\partial}{\partial y}$. If the vector field F_ε has only nondegenerate singularities and $\varepsilon \in (\mathbb{C}^2, 0)$ is sufficiently small, then the number of these singular points is exactly equal to the multiplicity by Theorem 8.21. By this definition, multiplicities of *nonsingular* points are equal to zero.

The definition of multiplicity does not depend on the choice of local coordinates used for writing the coefficients of the form. This follows from the deformational interpretation of the multiplicity. An alternative argument is as follows: changing the coordinates results in replacing the coefficients (f, g) of the form by another tuple of functions (f', g') belonging to the same ideal $\langle f, g\rangle$. If the change of coordinates is invertible, the two ideals are equal and so are the local algebras.

Our immediate goal is to compare the total multiplicity of all singularities of a foliation $\mathcal{F}$ and its blow-up $\widetilde{\mathcal{F}} = \pi^*\mathcal{F}$ for a simple blow-up π. Clearly, it is sufficient to consider the case where $\mathcal{F}$ has an isolated singularity on $(\mathbb{C}^2, 0)$ and the blow-up is the standard monoidal transformation $\sigma\colon (\mathbb{M}, \mathbb{E}) \to (\mathbb{C}^2, 0)$. The answer is different in the dicritical and nondicritical cases.

Consider the singular foliation $\mathcal{F}$ determined by 1-form $\omega = f\,dx + g\,dy$ of order n as in (8.27) and denote $\widetilde{\mathcal{F}}$ its blow-up as defined in Definition 8.11.

Theorem 8.33. *Let $\mathcal{F}$ be a singular foliation on $(\mathbb{C}^2, 0)$ and $\widetilde{\mathcal{F}}$ its blow-up. Then in all cases except for the dicritical singularity of order 1,*

$$\sum_{a\in S} \mu_a(\widetilde{\mathcal{F}}) = \mu_0(\mathcal{F}) - k(k-2) + n. \tag{8.28}$$

Here $n = \operatorname{ord}_0 \omega$, $m = \operatorname{ord}_0(xf + yg) \geqslant n+1$ (with the equality occurring in the nondicritical case) and

$$k = \min(n+2, m) = \begin{cases} n+1, & \text{in the nondicritical case,} \\ n+2, & \text{in the dicritical case.} \end{cases} \tag{8.29}$$

In the nondicritical case the formula (8.28) implies

$$\sum_{a} \mu_a(\widetilde{\mathcal{F}}) = \mu_0(\mathcal{F}) - (n^2 - n - 1). \tag{8.30}$$

In the dicritical case of order $n > 1$ the formula (8.28) yields

$$\sum_{a} \mu_a(\widetilde{\mathcal{F}}) = \mu_0(\mathcal{F}) - (n^2 + n). \tag{8.31}$$

In the dicritical case of order $n = 1$ we have $\mu_0(\mathcal{F}) = 1$ whereas the blow-up foliation $\widetilde{\mathcal{F}}$ is nonsingular, therefore

$$\sum_{a} \mu_a(\widetilde{\mathcal{F}}) = 0 = 1 - 1 = \mu_0(\mathcal{F}) - n^2. \tag{8.32}$$

Corollary 8.34. *If $n > 1$, then the total number of singularities of $\widetilde{\mathcal{F}}$ counted with their multiplicities, hence the multiplicity of every particular singularity, is strictly smaller than the multiplicity of the initial singularity,*

$$\sum_{a\in S} \mu_a(\widetilde{\mathcal{F}}) < \mu_0(\mathcal{F}). \quad \square \tag{8.33}$$

Proof of Theorem 8.33. We start with a convenient choice of the affine chart to work in. Making an affine transformation if necessary, we will then be able to assume without loss of generality that this chart is the standard affine chart U_1 with the coordinates (x, z).

First, we can assume that the only point *not* covered by the affine chart, is nonsingular for the blow-up foliation $\widetilde{\mathcal{F}}$. In the nondicritical case this is equivalent to assuming that the principal homogeneous part $h_{n+1} = xf_n + yg_n$ is not divisible by x.

Moreover, we can always assume in addition that the intersection of the divisors D_g and D_h is isolated: since $h = xf + yg$, this happens if and only if g is *not divisible* by x. To ensure that, we will assume that g_n is *not* divisible by x. Unlike the previous assumptions which can always be achieved by a suitable affine transformation, this last assumption can be achieved in all cases *except* for the dicritical case of order $n = 1$. In the latter case we always have $g_1(x, y) = x$ since the linear part of the corresponding vector field is a scalar matrix which remains scalar in any affine coordinates.

In the affine chart $U_1 \cong \mathbb{C}^2$ with the coordinates (x, z) the pullback of the form ω by the monoidal map $\sigma\colon (x, z) \mapsto (x, xz)$ was computed in (8.5). Technically it is more convenient to pull back the form $x\omega \in \Lambda^1(\mathbb{C}^2, 0)$: the fact that it has a nonisolated singularity does not matter, as the pullback will be in any case divided by a suitable power of x when extended on the exceptional divisor. The advantage is that the coefficients of the 1-*form* $\sigma_1^*(x\omega) = (\sigma_1^* h)\, dx + \sigma_1^*(x^2 g)\, dz$ are pullbacks of two *holomorphic germs* h and $g' = x^2 g$.

To extend the form $\sigma_1^*(x\omega)$ on the exceptional divisor $\mathbb{E} = \{x = 0\}$, one has to divide the coefficients $\sigma_1^* h$ and $\sigma_1^* g'$ by the maximal positive power x^k of the function x which is the local (relative to the chart U_1) equation of the exceptional divisor. Depending on whether the initial singularity is dicritical or not, we have two possibilities for this maximal order k, given by (8.29). The intersection multiplicity between $x^{-k}\sigma_1^* h$ and $x^{-k}\sigma_1^* g'$ at any point on the line $x = 0$ will then be the multiplicity of the corresponding singularity of the blow-up foliation.

On the language of the divisors the total multiplicity of all singular points of $\widetilde{\mathcal{F}}$ on the exceptional divisor $\mathbb{E}$ reduces to computation of the intersection index between the divisors $\sigma^{-1}(D_h) - k\mathbb{E}$ and $\sigma^{-1}(D_{x^2 g}) - k\mathbb{E} = \sigma^{-1}(D_g) - (k-2)\mathbb{E}$ in the open domain $U_1 \subset \mathbb{M}$. However, by our assumption that the point not covered by U_1 is nonsingular, we may extend the summation over all singular points on $\mathbb{E}$ using bilinearity and the rules established in Theorem 8.29:

$$\begin{aligned}\sum_a \mu_a(\widetilde{\mathcal{F}}) &= (\sigma^{-1}(D_h) - k\mathbb{E}) \cdot (\sigma^{-1}(D_{x^2 g}) - k\mathbb{E}) \\ &= (\sigma^{-1}(D_h) - k\mathbb{E}) \cdot \bigl(\sigma^{-1}(D_g) - (k-2)\mathbb{E}\bigr) \\ &= \sigma^{-1}(D_h) \cdot \sigma^{-1}(D_g) + k(k-2)\, \mathbb{E} \cdot \mathbb{E} \\ &= D_h \cdot D_g - k(k-2).\end{aligned} \tag{8.34}$$

It remains to compute the intersection index between two divisors $D_h, D_g \subset (\mathbb{C}^2, 0)$ at the origin, where $h = xf + yg$, and express it via $D_f \cdot D_g$. Using the fact that the intersection multiplicity depends only on the ideal generated by these germs, we obtain

$$D_h \mathbin{\overset{\circ}{\cdot}} D_g = D_{xf+yg} \mathbin{\overset{\circ}{\cdot}} D_g = D_{xf} \mathbin{\overset{\circ}{\cdot}} D_g = D_x \mathbin{\overset{\circ}{\cdot}} D_g + D_f \mathbin{\overset{\circ}{\cdot}} D_g.$$

The multiplicity of intersection $D_x \overset{\circ}{\cdot} D_g$ is equal to the order of the function $\operatorname{ord}_0 g(0,y)$. If g_n is not divisible by x, this order is equal to n, so that ultimately

$$D_h \overset{\circ}{\cdot} D_g = \mu_0(\mathcal{F}) + n, \qquad n = \operatorname{ord}_0 \mathcal{F}.$$

Putting everything together, we obtain the formula (8.28). □

8J. Desingularization of cuspidal points. Multiplicity of isolated singularities of order $n > 1$ goes down after blow-up (dicritical or not). To prove the desingularization theorems, we need to show that the only nonelementary points of order 1, the cuspidal points, can be desingularized in finitely many steps. Note that since the order of a cuspidal point is 1, the total multiplicity of all singularities which appear after blow-up (nondicritical) goes up by 1 by (8.30). We will show that for cuspidal points the multiplicity decreases after *two* consecutive blow-ups if it was three or higher, whereas a cusp of multiplicity 2 after three blow-ups gets desingularized into elementary points.

Without loss of generality we may assume that the lower order terms of the form ω are brought to the normal form

$$\begin{gathered} \omega = y\,dy + [f(x) + yg(x)]\,dx, \qquad f,g \in \mathbb{C}[[x]], \\ \operatorname{ord}_0 f = \mu \geqslant 2, \qquad \operatorname{ord}_0 g > 0. \end{gathered} \tag{8.35}$$

(cf. with (4.18)). In fact, we need only terms of order 2 for the analysis below. The number $\mu \geqslant 2$ is the multiplicity of the singular point (8.35).

The quadratic tangent form $xf_1 + yg_1$ for (8.35) is equal to y^2. It is nonzero (hence the singularity is nondicritical) and the only singular point after blow-up is the point $z = 0$ in the chart U_1, where the blow-up of ω takes the form

$$xz\,dz + (ax + bx^2 + cxz + z^2)\,dx + \mathfrak{m}^3 \otimes \Lambda^1, \tag{8.36}$$

where a, b are the leading coefficients of $f(x) = ax^2 + bx^3 + \cdots$ ($a \neq 0$ if and only if $\mu = 2$) and c the leading coefficient of $g(x) = cx + \cdots$. Here and below the notation $\mathfrak{m}^k$ is used to denote a collection of terms of order $\geqslant k$ and the tensor product stands for the 1-form with third order coefficients.

Further arguments are different for *simple cusp* with $\mu = 2$ and *higher cusps* with $\mu > 2$.

8**J**$_1$. *Simple cusp.* We show that after three consecutive blow-ups the simple cusp of multiplicity $\mu = 2$ can be blown up into three nondegenerate singularities.

If $\mu = 2$, then without loss of generality one may assume that $a = 1$. The order of the singularity (8.36) which appears after the first blow-up, is again 1 so it is a simple cusp, its multiplicity (by (8.30) with $n = 1$) is $3 = 2 + 1$ and the tangent form is $x^2 \not\equiv 0$. After the *second* blow-up (substitution

$x = uz$ and division by z) the cuspidal singular point (8.36) is transformed into the foliation defined by the form

$$uz\,dz + (u+z)(u\,dz + z\,du) + \mathfrak{m}^3 \otimes \Lambda^1, \tag{8.37}$$

which has a unique singularity at $u = 0$. The order of this singularity is now 2 and multiplicity is equal to $4 = 3 + 1$ by (8.30) (again with $n = 1$).

The tangent form for (8.37), $uz^2 + 2uz(u + z) = uz(2u + 3z)$, is the product of three different (simple) linear factors which means that after the *third* blow-up the foliation will have three singular points of total multiplicity $3 = 4 - 1$ (again by (8.30) yet this time with $n = 2$). This leaves only one combination of multiplicities $1, 1$ and 1 respectively, meaning that all three points are nondegenerate (hence elementary). One can show by direct computation that all three points are resonant saddles.

8$\mathbf{J}_2$. *Higher cusp.* In this case already after the first blow-up the form (8.36) has order 2, multiplicity $\mu + 1$ by (8.30) and the tangent form $xz^2 + x(bx^2 + cxz + z^2) = x(bx^2 + cxz + 2z^2)$ which is divisible by x but not a power of x. In other words, after the *second* blow-up there will appear at least *two* distinct points (three if $c^2 \neq 8b$) of *total* multiplicity μ by (8.30). This means that each of these two points has multiplicity at most $\mu - 1$ after *two* consecutive blow-ups.

Proof of Desingularization Theorems 8.14 and 8.15. We construct a sequence of blow-ups that would resolve completely an isolated singularity. The algorithm is very simple: starting from the initial singularity of a foliation $\mathcal{F} = \mathcal{F}_0$ at the origin $0 \in M_0 \cong (\mathbb{C}^2, 0)$, we construct a simultaneous simple blow-up $\pi_k \colon M_k \to M_{k-1}$, $k = 1, 2, \dots$, of all *nonelementary* singular points $\Sigma_{k-1} \subset M_{k-1}$ of the foliation $\mathcal{F}_{k-1}$ obtained on the previously constructed surface M_{k-1}.

The assertion on the vanishing divisor D (preimage of the origin) can be easily verified inductively. If $\gamma \subset M$ is a nonsingular curve biholomorphically equivalent to $\mathbb{P}^1$ and $a \in \gamma$ a center of blow-up $\pi \colon M' \to M$, then by Example 8.31 the blow-up $\pi^*\gamma$ will again be a nonsingular curve $\widetilde{\gamma}$ biholomorphically equivalent to γ and therefore again equivalent to $\mathbb{P}^1$ (note that the topology of embedding of $\widetilde{\gamma}$ in M' may change). If γ, γ' intersect transversally, then their blow-ups will be disjoint and both transversal to the exceptional divisor $\pi^{-1}(a) \subset M'$ created by π. Thus the assertion on the vanishing divisor reproduces itself inductively and holds at any moment. The proof of Theorem 8.14 is complete.

To prove Theorem 8.15, it remains to estimate the number of simple blow-ups before the algorithm terminates, i.e., before all singularities become elementary. Note that all singularities appearing in the process, can be organized in a tree graph with branches connecting each singularity with its descendants appearing by the simple blow-up. Take the longest branch in this tree, $0 = a_0$, $a_1 \in \Sigma_1$, $a_2 \in \Sigma_2$, *etc.* We claim that, with the possible exception of the last three steps, the multiplicity of singularities a_i decreases at least by one every step or, at worst, every two steps. Denoting by μ_i the respective multiplicities, we already know that:

(1) if a_i is of order > 1, then $\mu_{i+1} < \mu_i$ by Corollary 8.34;
(2) if a_i is of order 1 and is neither elementary nor simple cusp, then $\mu_{i+2} < \mu_i$ by §8$\mathbf{J}_2$;
(3) if a_i is a simple cusp, then the branch terminates after three more steps by §8$\mathbf{J}_1$.

These inequalities constrain the maximal length of the branch by $2(\mu - 1) + 3 = 2\mu + 1$. The proof of Theorems 8.14 and 8.15 is complete. □

8K. Concluding remarks: elimination of resonant nodes and dicritical tangencies. Elementary singular points can also be to some extent simplified by blow-up. For instance, a nondegenerate singularity with the eigenvalues λ_1, λ_2, defined by the Pfaffian equation

$$x\,dy + \lambda y\,dx + \cdots = 0, \qquad \lambda = -\lambda_1/\lambda_2 \neq -1,$$

is "split" by the blow-up into two singularities which are both nondegenerate when $\lambda \neq -1$. The corresponding negative ratios of eigenvalues will be $\lambda + 1$ and $(\lambda^{-1} + 1)^{-1}$.

The case $\lambda = -1$ corresponds either to the dicritical node $x\,dy + y\,dx + \cdots = 0$ or to the Jordan node $(x + y)\,dy + y\,dx + \cdots = 0$. The former singularity *disappears* after blow-up, while the latter produces an elementary singular point whose hyperbolic eigenspace is *transversal* to the exceptional divisor (the corresponding tangent form is y^2).

Combining these observations, one can make additional blow-ups on top of the desingularization achieved in Theorem 8.14 and *eliminate all resonant nodes with natural ratios of eigenvalues*. Indeed, such points correspond to negative natural values $\lambda = -n$ which can be increased by 1 in $n - 1$ steps until the parameter λ reaches the threshold value $\lambda = -1$ (all other singularities appearing in the process will be resonant saddles with $\lambda = n/(n-1)$). On the next step the singularity either disappears or becomes a saddle-node.

In another development, one can refine the assertion of the Desingularization Theorem 8.15 to *eliminate tangency points* between the foliation $\pi^*\mathcal{F}$ and the vanishing divisor D. We briefly outline here the required adjustments.

The tangency order between two smooth curves $\{f = 0\}$ and $\{g = 0\}$ is by definition the multiplicity of intersection $D_f \overset{a}{\bullet} D_g$ minus 1: if two curves intersect transversally, the tangency order is 0, for a true tangency it is always positive.

The *tangency order* between a foliation $\mathcal{F}$ defined by the Pfaffian equation $\omega = 0$ and a *smooth* analytic curve $\gamma = \{f = 0\}$ at a point a is defined only when γ is *not* a leaf or separatrix of $\mathcal{F}$.

If a is nonsingular for $\mathcal{F}$, then the tangency order $\tau_a(\mathcal{F}, \gamma)$ is by definition the tangency order between γ and the leaf of $\mathcal{F}$ passing through a. If γ is defined by the equation $\{f = 0\}$ locally near a, then one can easily verify that

$$\tau_a(\mathcal{F}, \gamma) = D_{\omega\wedge df} \overset{a}{:} D_f, \tag{8.38}$$

where $D_{\omega\wedge df}$ is the divisor of zeros of the 2-form $\omega \wedge df = \rho(x, y)\, dx \wedge dy$ identified with its coefficient ρ, $D_{\omega\wedge df} = D_\rho$.

Indeed, if the tangency order is k, then after choosing suitable local coordinates one can assume that $\omega = dy$ (recall that a is nonsingular) and $\gamma = \{f = 0\}$, $f(x, y) = y - b(x)$, $\operatorname{ord}_0 b = k + 1$. The expression in the right hand side of (8.38) will be then equal to the order of $\sigma(x, y) = db(x)/dx$ restricted on the smooth curve γ parameterized by x, i.e., to $k = \operatorname{ord}_0 b - 1$.

In the case where a is a singular point, one can use (8.38) as a *definition* of the tangency order. The important property of the tangency order thus defined, is the following one.

Proposition 8.35. *If a is a hyperbolic singular point of $\mathcal{F}$ which is not a resonant node, and L is a* separatrix *of the foliation $\mathcal{F}$ passing through it, then the order of tangency between L and any other smooth curve γ is by* 1 *greater than the order of tangency between $\mathcal{F}$ and γ,*

$$\gamma \overset{0}{:} L = \tau(\mathcal{F}, \gamma) + 1.$$

Proof. We can assume that the local coordinates are chosen so that the separatrix L is a coordinate axis, $L = \{y = 0\}$. Then $\omega = \lambda y(1 + \mathfrak{m})\, dx + (x + \mathfrak{m}^2)\, dy$, where λ is the negative ratio of eigenvalues.

A curve γ tangent to $\{y = 0\}$ with order $k \geqslant 0$, is defined by the equation $y - b(x) = 0$, $\operatorname{ord}_0 b = k + 1$. Direct computation of (8.38) yields

$$\tau_0(\mathcal{F}, \gamma) = \operatorname{ord}_{x=0}[\lambda b(x)(1 + \mathfrak{m}) - b'(x)(x + \mathfrak{m}^2)] = k + 1$$

if $\lambda \neq k + 1$, i.e., if the singular point is not a resonant node with the ratio of eigenvalues $-1 : (k + 1)$. □

Using the tangency order, one can combine the equalities (8.31) and (8.32) into a single identity valid for both $n > 1$ and $n = 1$. Assume that the origin is a *dicritical* singularity of a holomorphic foliation $\mathcal{F}$. Denote by Σ the singular locus of its blow-up $\widetilde{\mathcal{F}}$ and by T the collection of the tangency points between $\widetilde{\mathcal{F}}$ and the exceptional divisor.

Proposition 8.36. *If the singularity is dicritical of any order $n \geqslant 1$, then*

$$\sum_{a\in\Sigma} \mu_a(\widetilde{\mathcal{F}}) + \sum_{b\in T} \tau_b(\widetilde{\mathcal{F}}, S) = \mu_0(\mathcal{F}) - n^2. \tag{8.39}$$

Proof. When $n > 1$, the equality (8.39) follows from (8.31) and the observation that the order of tangency between $\widetilde{\mathcal{F}}$ given by the Pfaffian equation $x^{-n}[(\cdots)\,dx + g(x, xz)\,dz]$ and $\mathbb{E} = \{x = 0\}$ at any point is equal to the order of the root of the function $x^{-n}g(x, xz) = g_n(1, z) + \cdots$ restricted on $\mathbb{E}$. The total multiplicity of all roots of $g_n(1, z)$ is equal to n, which proves (8.39) for $n > 1$. For $n = 1$ this formula is proved by direct inspection: there are neither singular nor tangency points after blow-up, whereas the initial multiplicity $\mu_0(\mathcal{F})$ is equal to 1. □

Behavior of tangency points after blow-up can be easily controlled: by (8.26), the intersection multiplicity between two *smooth* analytic curves decreases by 1 after blow-up. Using this fact, one can achieve by elementary inductive arguments the following improvement of the Desingularization Theorem 8.14.

Theorem 8.37 ([**Kle95**]). *In the formulation of the Desingularization Theorem 8.14 one can always guarantee that the dicritical components of the vanishing divisor $D = \pi^{-1}(0)$ carry no tangency points with the foliation $\pi^*\mathcal{F}$ (in particular, no singularities of the latter).*

The number of simple blow-ups necessary to desingularize the singular point of multiplicity μ in this strong sense does not exceed $\mu + 2$.

Exercises and Problems for §8.

Exercise 8.1. Compute blow-ups of:

(1) a smooth analytic curve passing through 0,
(2) several lines through 0 crossing each other by nonzero angles,
(3) the cusp $y^2 - x^3 = 0$.

Exercise 8.2. What happens after blow-up of a *nonsingular* point of a vector field?

Exercise 8.3. What happens after blow-up of a *homogeneous* vector field?

Problem 8.4. Give direct algebraic proof of Proposition 8.25 based on constructing the basis for the local algebra $Q_{fg,h}$ from the bases of the local algebras $Q_{f,h}$ and $Q_{g,h}$ respectively.

Exercise 8.5. Compute the ratios of eigenvalues for all three nondegenerate singular points obtained by complete desingularization of the simple cuspidal point described in §8$\mathbf{J}_1$.

Problem 8.6. Prove that any holomorphic vector field $F = (F_1, F_2)$ with an *isolated* singular point at the origin $0 \in \mathbb{C}^2$ satisfies the *Lojasiewicz condition*: there exist finite positive C and M such that $|F(x)| > C|x|^M$ for all $x \in (\mathbb{C}^2, 0) \smallsetminus \{0\}$.

Problem 8.7. Prove that consecutive desingularization of a *rational node*, a singularity with the ratio of eigenvalues $\lambda = p/q \in \mathbb{Q}$, $p, q \neq 1$, necessarily involves a dicritical blow-up on some step. How many standard simple blow-ups are required to obtain a singular point whose subsequent blow-up is dicritical?

Problem 8.8. Suppose that the complete desingularization of an isolated singularity of multiplicity μ does not involve neither cusps nor dicritical blow-ups. Give an upper bound for the number of blow-ups in the desingularization, better than in Theorem 8.15.

Problem 8.9. Suppose that a nice blowing up of an isolated singular point of a planar analytic vector field is completely nondicritical and has at most one noncorner singular point. Prove that all the characteristic numbers (ratios of the eigenvalues) of the singular points of the nice blowing up are rational.