Exam for winter semester 2007/8

Rules of the game This exam is intended mostly for the students who took the reading course "Analytic and Geometric Theory of Ordinary Differential Equations", based on the weblog at `http://yakovenko.wordpress.com` and the textbook by Yu. Ilyashenko and S. Yakovenko "Lectures on Analytic Differential Equations", AMS Publ., 2007, cited therein. However, everybody can submit the exam.

The exam is take-home and has to be submitted by February 28, 2008 (i.e., *two weeks after the official end* of the winter semester). Of course, earlier submissions are welcome, but the deadline will be observed strictly.

Unless explicitly stated otherwise, you may help yourself with any results from the textbook, provided that they are accurately cited.

Problem I.1. Give a complete analytic classification of the holomorphic flows on the Riemann sphere $\mathbb{P}^1$ (i.e., construct a list, finite or infinite, of flows such that every holomorphic flow in analytically equivalent to one of the flows from the list, while any two different flows in the list are *not* holomorphically equivalent.

Problem I.2. Prove that the constant holomorphic vector fields $\frac{\partial}{\partial z}$ on the two tori $\mathbb{T}_1 = \mathbb{C}/(\mathbb{Z} + i\mathbb{Z})$ and $\mathbb{T}_2 = \mathbb{C}/(\mathbb{Z} + 2i\mathbb{Z})$, are not holomorphically equivalent.

Problem I.3. Prove that each holonomy operator g corresponding to any separatrix $S \subset (\mathbb{C}^2, 0)$ of an integrable foliation $du = 0$ with an analytic potential $u \in \mathcal{O}(\mathbb{C}^2, 0)$, is periodic: some iterated power of g is identity.

Problem I.4. Consider the Riccati equation

$$\frac{dz}{dt} = a(t)\, z^2 + b(t)\, z + c(t), \qquad a, b, c \in \mathcal{M}(\mathbb{P}) \cong \mathbb{C}(t), \tag{0.1}$$

with meromorphic coefficients a, b, c having poles only on the finite point set $\Sigma \subseteq \mathbb{P}$. Is it true that solutions of this equation can be continued along any path on the t-plane, avoiding the singular locus Σ?

Prove that equation (0.1) defines a singular holomorphic foliation $\mathcal{F}$ on the compactified phase space $\mathbb{P}^1 \times \mathbb{P}^1$, which is transversal to any "vertical" projective line $\{t = a\}$, $a \notin \Sigma$. Show that each leaf of $\mathcal{F}$ can be continued over any path in the t-sphere, avoiding the singular locus. Prove that the induced transformation between any two cross-sections $\{t = a\} \times \mathbb{P}^1$ and $\{t = b\} \times \mathbb{P}^1$, $a, b \notin U$, is a well-defined Möbius transformation (fractional linear map $z \mapsto \frac{\alpha z + \beta}{\gamma z + \delta}$ with $\alpha\delta - \beta\gamma \neq 0$). Does $\mathcal{F}$ always possess a separatrix?

Problem I.5. How many separatrices a *homogeneous* vector field of degree r on $\mathbb{C}^2$ may have? How many separatrices a *generic* homogeneous vector field has?

Problem I.6. Find *all* complex logarithms of the matrix $M = \begin{pmatrix} -1 & 1 \\ & -1 \end{pmatrix}$ (i.e., solutions of the equation $\exp A = M$).

Problem I.7. Describe the *real* formal normal forms for vector fields in $\mathbb{R}^3$ with the spectrum $0, \pm i\omega$, $\omega > 0$.

Problem I.8. Prove that the formal normal form of a vector field in the Poincaré domain in $\mathbb{C}^n$ is integrable in quadratures for any n.

Problem I.9 (alternative proof the Poincaré linearization theorem)**.** Let $f \in \operatorname{Diff}(\mathbb{C}, 0)$ be a contracting hyperbolic holomorphic self-map, $f(z) = \lambda z + \cdots$, $|\lambda| < 1$, and $g(z) = \lambda z$ its linearization (the normal form).

Prove, *without using the Poincaré linearization theorem*, that the sequence of iterations $h_n = g^{-\circ n} \circ f^{\circ n}$ is defined and converges in some small disk around the origin. Prove that the limit $h = \lim h_n$ conjugates f and g.

Problem I.10 (formal rigidity of generic groups)**.** Assume that two finitely generated subgroups $G, G' \subseteq \operatorname{Diff}(\mathbb{C}, 0)$ of conformal germs are *formally* equivalent and one of these groups contains a hyperbolic germ. Prove that in such case G and G' are holomorphically equivalent, moreover, any formal conjugacy between them is necessarily holomorphic (convergent).

Problem I.11. Prove that any holomorphic vector field $F = (F_1, F_2)$ with an *isolated* singular point at the origin $0 \in \mathbb{C}^2$ satisfies the *Lojasiewicz* *condition*: there exist finite positive C and M such that $|F(x)| > C|x|^M$ for all $x \in (\mathbb{C}^2, 0) \smallsetminus \{0\}$.

Problem I.12. Prove that consecutive desingularization of a *rational node*, a singularity with the ratio of eigenvalues $\lambda = p/q \in \mathbb{Q}$, $p, q \neq 1$, necessarily involves a dicritical blow-up on some step. How many standard simple blow-ups are required to obtain a singular point whose subsequent blow-up is dicritical?