

H. Find the expression of the corresponding element $[\Gamma_{-\varepsilon}] \in H_1(L_{-\varepsilon}, \mathbb{Z})$ through $[\Gamma_{-\varepsilon}^1], [\Gamma_{-\varepsilon}^2], [\delta_{-\varepsilon}]$.

Problem 26.24. Prove that for any r there exists a real polynomial vector field from the class \mathcal{A}_r with a limit cycle of multiplicity $\frac{1}{2}(r+1)(r-2)$.

Problem 26.25. Modifying the proof of Theorem 26.35, write explicitly the Picard–Fuchs system for the cubic polynomial $f(x, y) = \frac{1}{2}(x^2 + y^2) - \frac{x^3}{3}$ and the two forms $\omega_1 = y dx$ and $\omega_2 = xy dx$.

Problem 26.26. Let $f(x, y) = y^2 + x^{n+1} + p_n(x)$ be a *hyperelliptic* polynomial. Prove that the forms $y dx, xy dx, \dots, x^{n-1}y dx$ form a basis of the corresponding Petrov module \mathbf{P}_f . Modifying the proof of Theorem 26.35, write explicitly the Picard–Fuchs system for the corresponding *hyperelliptic integrals*.

27. Topological classification of complex linear foliations

The famous Grobman–Hartman theorem [Gro62, Har82] asserts that any real vector field whose linearization matrix is hyperbolic (i.e., has no eigenvalues with zero real part), is topologically orbitally equivalent to its linearization. An elementary analysis shows that two hyperbolic linear real vector fields are orbitally topologically conjugated if and only if they have the same number of eigenvalues to both sides of the imaginary axis.

This section describes the complex counterparts of these results. From the real point of view a holomorphic 1-dimensional singular foliation on $(\mathbb{C}^n, 0)$ by phase curves of a holomorphic vector field is a 2-dimensional real analytic foliation on $(\mathbb{R}^{2n}, 0)$. If the singularity at the origin is in the Poincaré domain, this foliation induces a *nonsingular real 1-dimensional foliation (trace)* on all small $(2n-1)$ -dimensional spheres $\mathbb{S}_\varepsilon^{2n-1} = \{|x_1|^2 + \dots + |x_n|^2 = \varepsilon > 0\}$. Under the *complex* hyperbolicity-type conditions excluding resonances, the trace is generically structurally stable. Poincaré resonances manifest themselves via *bifurcations* of this trace foliation.

On the contrary, if the singularity is in the Siegel domain, the corresponding foliations exhibit *rigidity*: two foliations are topologically equivalent if and only if there is a rather special conjugacy between them which is completely determined by n complex numbers. This rigidity implies that there are *continuous invariants* (moduli) of topological classification.

27A. Trace of the foliation on the small sphere. Consider the real sphere of radius $\varepsilon > 0$,

$$\mathbb{S}_r = \{r^2(x) = \varepsilon\} \subseteq \mathbb{C}^n, \quad r^2(x) = |x|^2 = \sum_1^n x_i \bar{x}_i. \quad (27.1)$$

The differential of the (nonholomorphic) function $r^2: \mathbb{C}^n \rightarrow \mathbb{R}$ is a complex-valued 1-form, $dr^2 = x d\bar{x} + \bar{x} dx$, which on the complex vector field $F(x) = (v_1(x), \dots, v_n(x))$ takes the value

$$dr^2 \cdot v(x) = \sum_{i=1}^n x_i \bar{v}_i + \bar{x}_i v_i = 2 \operatorname{Re} \left(\sum x_i \bar{v}_i \right) \in \mathbb{R}.$$

If $F(x) = \Lambda x$ is a linear diagonal vector field with the eigenvalues $\lambda_1, \dots, \lambda_n \in \mathbb{C}$, then

$$dr^2 \cdot F = 2 \operatorname{Re} \sum \lambda_i |x_i|^2.$$

The following observation due to V. Arnold [Arn69] gives a topological characterization of the Poincaré type holomorphic foliations.

Proposition 27.1. *All complex phase curves of the diagonal linear vector field Λx of Poincaré type in \mathbb{C}^n are transversal as 2-dimensional embedded surfaces, to all spheres \mathbb{S}_ε , $\varepsilon > 0$.*

Proof. The tangent space to any trajectory considered as a real 2-dimensional surface in $\mathbb{R}^{2n} = \mathbb{C}^n$, is spanned over \mathbb{R} by the vectors $v(x) = \Lambda x$ and $v'(x) = i\Lambda x$. To prove the transversality, it is sufficient to verify that the 1-form dr^2 cannot vanish on both vectors simultaneously for $x \neq 0$.

If the spectrum belongs to the Poincaré domain, then without loss of generality we may assume that

$$\operatorname{Re} \lambda_i < 0, \quad i = 1, \dots, n. \quad (27.2)$$

Indeed, replacing the field Λx by the orbitally equivalent field $\alpha \Lambda x$, $|\alpha| = 1$, preserves all holomorphic phase curves but rotates the spectrum of Λ as a whole.

Under the assumption (27.2) the expression

$$dr^2 \cdot F = s(x) = \sum \lambda_i |x_i|^2 \in \mathbb{C} \quad (27.3)$$

is in the left half-plane, moreover,

$$\operatorname{Re} s(x) \leq \delta |x|^2 < 0, \quad \delta > 0. \quad (27.4)$$

This implies the required transversality. \square

Remark 27.2. Transversality is an open condition: sufficiently small perturbations of the vector field leave it transversal to the compact sphere.

In particular, if $F(x) = \Lambda x + w(x)$ is a nonlinear vector field, then the rescaling $x \mapsto \varepsilon x$ conjugates its restriction on the ε -sphere $\mathbb{S}_\varepsilon^{2n-1}$ with the restriction of the field $F_\varepsilon(x) = \Lambda x + \varepsilon^{-1}w(\varepsilon x)$ on the unit sphere \mathbb{S}_1^{2n-1} . But since the nonlinear part $w(x)$ is at least of second order, the field F_ε is ε -uniformly close on the unit sphere to the linear field $F_0(x) = \Lambda x$. Thus we

conclude that the *nonlinear* vector field F is transversal to all sufficiently small spheres $\mathbb{S}_\varepsilon^{2n-1}$.

Definition 27.3. Let $\mathcal{F} = \{L_\alpha\}$ be a foliation on a manifold M . The *trace of the foliation* on a submanifold $N \subset M$ is the partition of N into connected components of intersection of the leaves L_α with N , $\mathcal{F}|_N = \{L_\alpha \cap N\}$.

In general, the trace of a foliation need not itself be a foliation; the intersections $L_\alpha \cap N$ can be nonsmooth in general. Even in the analytic context one cannot exclude the appearance of singularities.

Corollary 27.4. *The trace of the holomorphic foliation \mathcal{F} induced by a linear vector field of Poincaré type on any sphere $\mathbb{S}_\varepsilon^{2n-1}$ is a smooth (actually, real analytic) nonsingular real 1-dimensional foliation $\mathcal{F}' = \mathcal{F}|_{\mathbb{S}_\varepsilon}$.*

Proof. By the implicit function theorem, intersection of each leaf with the sphere is a smooth curve. □

Moreover, for singularities of Poincaré type the trace of the foliation on a (sufficiently small) sphere determines completely the foliation up to the topological equivalence, even if the vector field spanning the foliation is nonlinear.

Definition 27.5. A (topological) *cone* over a set $K \subset \mathbb{C}^n \setminus \{0\}$ is the set $\mathcal{C}K = \{rx : 0 \leq r \leq 1, x \in K\} \subseteq \mathbb{C}^n$. If \mathcal{F}' is a foliation on the sphere $\mathbb{S}_1^{2n-1} \subset \mathbb{C}^n$, then the *cone over the foliation* $\mathcal{C}\mathcal{F}'$ is the foliation of $\mathbb{C}^n \setminus \{0\}$ whose leaves are the cones over the leaves of \mathcal{F}' .

Theorem 27.6. *A singular foliation \mathcal{F} on $(\mathbb{C}^n, 0)$, generated by a vector field of Poincaré type, is topologically equivalent to the cone over its trace $\mathcal{F}'_\varepsilon = \mathcal{F}|_{\mathbb{S}_\varepsilon^{2n-1}}$ on any sufficiently small sphere.*

Proof. Under the normalizing assumption (27.2) the real flow of the vector field Λx , the one-parametric subgroup of linear maps $\{\Phi^t = \exp t\Lambda : t \in \mathbb{R}\}$ is locally contracting: orbits $\Phi^t(x)$, $x \in \mathbb{S}_1^{2n-1}$ of all points uniformly converge to the origin as $t \rightarrow +\infty$. This follows again from (27.4): if ε is so small that $|w(x)| < \frac{\delta}{2}|x|$ for $|x| < \varepsilon$, we have $|\Phi^t(x)| < \exp(-\delta t/4)|x|$ for all $t > 0$.

The real flow Φ^t is tangent to the foliation \mathcal{F} . Thus the map h of the small ε -ball $\{|x| \leq \varepsilon\}$ into itself, defined by the formulas

$$h(rx) = \Phi^{-\ln r}(x), \quad 0 < r \leq 1, \quad x \in \mathbb{S}_\varepsilon^{2n-1}, \quad h(0) = 0,$$

is a homeomorphism conjugating $\mathcal{C}(\mathcal{F}|_{\mathbb{S}_\varepsilon})$ with \mathcal{F} . □

In particular, Theorem 27.6 implies that all foliations \mathcal{F}'_ε are topologically equivalent to each other. Yet without the additional assumptions they may be nonequivalent to the foliation \mathcal{F}'_0 which is the trace of the linear

foliation \mathcal{F}_0 on (any) sphere. This additional assumption is called *complex hyperbolicity*.

27B. Structural stability of the trace of hyperbolic foliation.

Definition 27.7. A holomorphic germ of a vector field $\dot{x} = Ax + \dots$ in $(\mathbb{C}^n, 0)$ is *complex hyperbolic* (or just *hyperbolic*⁷ if this does not lead to confusion), if no two eigenvalues λ_i, λ_j of the linearization matrix A differ by a real factor,

$$\lambda_i/\lambda_j \notin \mathbb{R} \quad \text{for all } i \neq j. \quad (27.5)$$

In particular, A must be nondegenerate and diagonalizable.

Under the additional assumption of complex hyperbolicity we can completely describe the trace of the linear diagonal foliation and show that it is structurally stable: any C^1 -small perturbation produces a foliation that is topologically equivalent to the initial one.

Everywhere below in this section \mathcal{F} is a singular foliation of \mathbb{C}^n by phase curves of the complex hyperbolic vector field Ax with the eigenvalues $\lambda_1, \dots, \lambda_n$ of the diagonal matrix A in the Poincaré domain and, if necessary, normalized by the condition (27.2). We denote by \mathcal{F}' its restriction on \mathbb{S}_1^{2n-1} .

The first immediate consequence of complex hyperbolicity is the fact that the only multiply-connected leaves of the foliation \mathcal{F} by complex phase curves of a diagonal linear system, are its separatrices.

Proposition 27.8. *The only multiply-connected leaves of a foliation generated by complex hyperbolic linear system $\dot{x} = Ax$ in \mathbb{C}^n are its separatrices which are lines spanned by the eigenvectors of A . All other leaves of \mathcal{F} are simply connected.*

Proof. Without loss of generality we may assume that A is diagonal, $A = \Lambda = \text{diag}\{\lambda_1, \dots, \lambda_n\}$. The application

$$t \mapsto x(t) = (c_1 \exp(\lambda_1 t), \dots, c_n \exp(\lambda_n t)) = \Phi^t(c), \quad c \in \mathbb{C}^n, \quad (27.6)$$

parameterizes the phase curve passing through a point $c \in \mathbb{C}^n$. This parametrization is not injective, if $\exp t\lambda_j = 1$ for some t and all j corresponding to nonzero coordinates of the point c . If there is only one such coordinate, then the noninjectivity is indeed possible if $t = 0 \pmod{T_j}$, where T_j is the corresponding period. If a has at least two nonzero coordinates j and k , then the simultaneous occurrence $t = 0 \pmod{T_j}$ and $t = 0 \pmod{T_k}$ is impossible: it would mean that the ratio T_j/T_k is rational hence real. \square

⁷In order to distinguish this from the *real hyperbolicity* of self-maps, introduced in Definition 7.2. The reasons why two seemingly different notions are called by similar names, are clarified by Proposition 27.10 below.

Assume that in addition to the normalizing condition (27.2), the enumeration of the eigenvalues $\lambda_1, \dots, \lambda_n$ is chosen in the increasing order of their arguments in the interval $[0, 2\pi)$,

$$\text{Arg } \lambda_1 < \text{Arg } \lambda_2 < \dots < \text{Arg } \lambda_{n-1} < \text{Arg } \lambda_n \tag{27.7}$$

(this is possible since by the hyperbolicity assumption $\lambda_j/\lambda_k \notin \mathbb{R}$, so all values of the arguments are distinct).

Since the coordinate axes are leaves of \mathcal{F} , the big circles $C_i = \{x_j = 0, j \neq i, |x_i| = 1\}$ are leaves of \mathcal{F}' . We show that all other leaves are bi-asymptotic to these circles.

Proposition 27.9. *If Λ is hyperbolic, then the limit set $\bar{\gamma} \setminus \gamma$ of any leaf $\gamma \in \mathcal{F}'$ different from C_j , is the union of two big circles $C_j \cup C_k, j \neq k$.*

Proof. Any leaf L_c of the “large” foliation \mathcal{F} passing through a point $c \in \mathbb{C}^n$ is parameterized by the map (27.6). The intersection $\gamma_c = L_c \cap \mathbb{S}_1^{2n-1}$ is defined by the equation

$$|c_1|^2 \exp 2 \text{Re}(\lambda_1 t) + \dots + |c_n|^2 \exp 2 \text{Re}(\lambda_n t) = 1. \tag{27.8}$$

As follows from the transversality property, this is a smooth curve parameterized by a smooth curve $\tilde{\gamma}_c$ on the t -plane, defined by the equation (27.8).

The curve $\tilde{\gamma}_c$ *a priori* may have compact and noncompact components. But any compact component must bound a compact set in $\mathbb{C} \cong L_c$ so that the function $|x(t)|$ has critical points inside. Such critical points correspond to nontransversal intersections that are forbidden by Proposition 27.1.

Thus γ_c may consist of only noncompact components (eventually, several) along which $|t|$ tends to infinity. But as $|t| \rightarrow \infty$, the growth rate of each exponential term $\exp(2 \text{Re}(\lambda_j t))$ is determined by the angular behavior of t . In particular, since all exponentials in (27.8) should be bounded (unless the corresponding coefficients c_j vanish), we have the necessary condition that all limit directions $\lim\{t/|t|: t \in \tilde{\gamma}_c, |t| \rightarrow +\infty\}$ must be within the sector $S_c = \bigcap_{j: c_j \neq 0} \{\text{Re } \lambda_j t \leq 0\}$. However, if t tends to infinity (asymptotically) *in the interior* of this sector, then all exponents will tend to zero in violation of (27.8).

Thus if L_c is *not* a separatrix (i.e., more than one coefficient c_j is nonzero), the curve $\tilde{\gamma}_c$ must be bi-asymptotic to the two boundary rays of the sector S_c . This in turn means that the corresponding trajectory γ_c is bi-asymptotic to the two cycles $C_j \neq C_k$. □

Behavior of leaves near each cycle C_j is determined by the iterations of the corresponding holonomy map of the foliation \mathcal{F}' which can be easily

expressed in terms of the holonomy of the corresponding complex separatrix $\mathbb{C}e_j$, $e_j = (0, \dots, \underset{j}{1}, \dots, 0) \in \mathbb{C}^n$, of the initial holomorphic foliation \mathcal{F} .

Consider the circular leaf $C_j \subset \mathbb{S}_1^{2n-1}$ of the foliation \mathcal{F}' with the orientation induced by the counterclockwise (positive) direction of going around the origin in the j th coordinate axis. Then for any (smooth) $(2n-2)$ -dimensional cross-section $\tau'_j: (\mathbb{R}^{2n-2}, 0) \rightarrow (\mathbb{S}_1^{2n-1}, e_j)$ transversal to the trace foliation \mathcal{F}' at the point $e_j \in C_j$, one can define the first return map (holonomy) $h_j = \Delta_{C_j}: (\tau'_j, 0) \rightarrow (\tau'_j, 0)$.

Proposition 27.10. *The holonomy $h_j \in \text{Diff}(\mathbb{R}^{2n-2}, 0)$ of each cycle C_j is differentiably conjugate to the diagonal linear map $A_j \in \text{Diff}(\mathbb{C}^{n-1}, 0)$ hyperbolic in the sense of Definition 7.2: its eigenvalues $\{2\pi i \lambda_k / \lambda_j\}$, $k \neq j$ are all off the unit circle.*

Proof. Since the sphere \mathbb{S}_1^{2n-1} is transversal to the foliation \mathcal{F} , any smooth (nonholomorphic) cross-section $\tau'_j: (\mathbb{R}^{2n-2}, 0) \rightarrow (\mathbb{S}_1^{n-1}, e_j)$ transversal to the trace foliation \mathcal{F}' at the point $e_j \in C_j$ inside \mathbb{S}_1^{2n-1} , will also be transversal to the complex separatrix of \mathcal{F} lying on the j th coordinate axis.

The holonomy maps for the foliation \mathcal{F} associated with the two cross-sections, τ'_j and the “standard” cross-section $\tau_j: (\mathbb{C}^{n-1}, 0) \rightarrow (\mathbb{C}^n, e_j)$, are smoothly conjugate, in fact, the conjugacy is real analytic as a germ between $(\mathbb{R}^{2n-2}, 0)$ and $(\mathbb{C}^{n-1}, 0)$. The holonomy for the “standard” cross-section was already computed; see Example 2.28. \square

Proposition 27.11. *The unstable (resp., stable) manifold of the cycle C_j is the sphere $\mathbb{S}_1^{j-1} = \{x_{j+1} = \dots = x_n = 0\} \cap \mathbb{S}_1^{2n-1}$ (resp., the sphere $\mathbb{S}_1^{n-j-1} = \{x_1 = \dots = x_{j-1} = 0\} \cap \mathbb{S}_1^{2n-1}$).*

Proof. The corresponding complex coordinate planes \mathbb{C}^{j-1} and \mathbb{C}^{n-j-1} in \mathbb{C}^n are invariant by the foliation \mathcal{F} and the computations of the preceding proof show that the restriction of the first return map on the corresponding spheres (in intersection with the cross-section τ'_j) has only eigenvalues $\exp 2\pi i \lambda_k / \lambda_j$. All these numbers are of modulus less than one (resp., greater than one). Since the stable (unstable) manifolds are uniquely defined, this proves the proposition. \square

The properties of the foliation \mathcal{F}' established by these three propositions, imply its *structural stability*: any sufficiently close foliation is topologically equivalent to \mathcal{F}' .

Theorem 27.12 (J. Guckenheimer, 1972 [Guc72]). *Assume that the diagonal matrix Λ is complex hyperbolic and in the Poincaré domain.*

Then the holomorphic vector field $F(x) = \Lambda x + w(x)$ is topologically orbitally linearizable, i.e., the holomorphic singular foliation of $(\mathbb{C}^n, 0)$ by complex phase curves of the holomorphic vector field is locally topologically equivalent to the foliation defined by the linear vector field $F_0(x) = \Lambda x$.

Moreover, any sufficiently close vector field is locally topologically orbitally equivalent to F .

Proof. Consider the rescaling $F_\varepsilon(x) = \varepsilon^{-1}F(\varepsilon x)$, the corresponding foliation \mathcal{F}_ε in the ball $\{|x| < 1\}$ and its trace \mathcal{F}'_ε on the unit sphere $\mathbb{S}_1^{2n-1} = \varepsilon^{-1}\mathbb{S}_1^{2n-1}$.

By Theorem 27.6, both foliations \mathcal{F}_ε and \mathcal{F}_0 are topological cones over their traces \mathcal{F}'_ε and \mathcal{F}'_0 . The assertion of the theorem will follow from the topological equivalence of the latter two foliations on \mathbb{S}_1^{2n-1} .

By the Palis–Smale theorem [PS70a], a vector field on the compact manifold is structurally stable (i.e., its phase portrait is topologically orbitally equivalent to that of any sufficiently C^k -close vector field) if it meets the following *Morse–Smale conditions*:

- (1) its singular points and limit cycles are hyperbolic (i.e., all eigenvalues of the linearization at any singular point have nonzero real parts, and all multipliers of any limit cycle have modulus different from 1);
- (2) its orbits can accumulate only to singular points or limit cycles;
- (3) all stable and unstable invariant manifolds of singular points and limit cycles (which exist by the hyperbolicity assumption) intersect transversally.

All these conditions for the foliation \mathcal{F}'_0 are verified in Propositions 27.10, 27.9 and 27.11 respectively. Therefore the foliation \mathcal{F}'_0 is structurally stable and hence topologically equivalent to \mathcal{F}'_ε for all small ε .

Returning to the initial nonlinear vector field $F = F_1$, we conclude that it is topologically orbitally equivalent to its linearization in all sufficiently small balls $\{|x| < \varepsilon\}$. \square

Corollary 27.13. *Any two linear complex hyperbolic vector fields of Poincaré type in \mathbb{C}^n generate globally topologically equivalent singular foliations.*

Any two nonlinear holomorphic vector fields in $(\mathbb{C}^n, 0)$, whose linearizations are complex hyperbolic vector fields of Poincaré type, generate locally topologically equivalent singular holomorphic foliations.

Proof. Since topological equivalence is transitive, by Theorem 27.12 the second assertion of the corollary follows from the first one.

To prove the assertion on linear systems, note that any two complex hyperbolic matrices in the Poincaré domain can be continuously deformed into each other within this class. Indeed, any such matrix can be first diagonalized and all its eigenvalues brought into the open left half-plane. Then all absolute values of these eigenvalues can be made equal to 1 without changing their arguments; this will affect neither hyperbolicity nor the Poincaré property. Finally, the arguments of the eigenvalues can be assigned any positions, say, at equal angles between $\pi/2$ and $-\pi/2$. In this normal form the two diagonal matrices of the same size differ only by reordering of the coordinate axes. \square

27C. Resonances in the Poincaré domain. Without complex hyperbolicity the foliation traced by a *linear* system on the unit sphere is still nonsingular, but may have nontrivial recurrence. Indeed, in this case the first return map for one of the cycles will have a multiplier $\exp 2\pi i \lambda_1 / \lambda_2$ which has modulus 1. The corresponding foliation \mathcal{F}' on the sphere will then have a family of invariant 2-tori foliated by periodic or quasiperiodic orbits, depending on whether the ratio $\lambda_1 / \lambda_2 \neq 1$ is rational or not. Since both rational and irrational numbers are dense, two nonhyperbolic linear systems in the Poincaré domain can be arbitrarily close to each other but topologically nonequivalent.

Generically, occurrence of multiple eigenvalues leads to the linearization matrix with a nontrivial Jordan normal form. Consider for simplicity the case $n = 2$, where such a form is necessarily a block of size 2. Then the corresponding foliation has only one complex separatrix. The same arguments as were used in the proof of Proposition 27.10 show that this separatrix leaves the trace in the form of a cycle on the sphere \mathbb{S}^3 whose first return map is conjugate to the complex holonomy of the separatrix.

Somewhat surprisingly and in contrast with the previously discussed diagonal cases, the holonomy map of this separatrix is *essentially nonlinear*: it cannot be linearized by a suitable choice of the cross-section (or, what is the same, a chart on it), as explained in Example 2.30. The computation below for the case where $n = 1$ shows that the holonomy has a fixed point of multiplicity exactly equal to 2 and thus a small perturbation will produce two close fixed points corresponding to two cycles of the trace foliation.

Occurrence of nonlinearities affects the situation in a similar way when (Poincaré) resonances occur, as was observed in [Arn69]. Consider the simplest Poincaré resonance in \mathbb{C}^2 and compute the holonomy map.

Proposition 27.14. *Consider a planar resonant singularity of the Poincaré type in the formal normal form*

$$\dot{x} = nx + ay^n, \quad \dot{y} = y, \quad a \in \mathbb{C}, \quad n \geq 1. \quad (27.9)$$

Then the holonomy Δ of the unique separatrix $y = 0$, computed for the standard cross-section $\tau = \{x = 1\}$, is tangent to a rotation by the rational angle $2\pi/n$ and its n th iteration has an isolated fixed point of multiplicity $n + 1$ at the origin.

Proof. The system (27.9) is integrable: its general solution is $y(t) = C \exp t$, $x = (C' + aC^n t) \exp nt$, with arbitrary constants $C, C' \in \mathbb{C}$. The initial condition $(x(0), y(0)) = (1, s) \in \tau$ yields for the corresponding solution the formula

$$x(t) = (1 + as^n t) \exp nt, \quad y(t) = s \exp t.$$

For $s = 0$ the x -component of the solution (separatrix) is $2\pi/n$ -periodic. For small $s \in (\mathbb{C}, 0)$, the solution with this initial condition crosses again the section τ at the moments $t_k(s) = 2\pi ik/n + \delta_k(s)$, $\delta_k(s) = o(1)$, $k = 1, 2, \dots$, where $\delta_k(s)$ is the complex root of the equation

$$1 + as^n(2\pi ik/n + \delta_k(s)) = \exp(-n\delta_k(s)) = 1 - n\delta_k(s) + \dots, \quad \lim_{s \rightarrow 0} \delta_k(s) = 0.$$

This equation can be resolved with respect to $\delta_k(s)$ defining the latter as an analytic function of s by the implicit function theorem. Computing the Taylor terms, we see immediately that

$$\delta_k(s) = -\frac{2\pi ika}{n^2} s^n + \dots, \quad t_k(s) = \frac{2\pi ik}{n} + \delta(s).$$

The iterated power of the holonomy map Δ^k is therefore

$$\Delta^k(s) = s \exp t_k(s) = \lambda^k s \exp \delta(s) = \lambda^k s(1 - kA s^n + \dots),$$

$$\lambda = \exp \frac{2\pi i}{n}, \quad A = \frac{2\pi ia}{n^2} \neq 0.$$

The n th iterated power of Δ is tangent to the identity and has an isolated fixed point of multiplicity exactly $n + 1$. □

Corollary 27.15. *The resonant node corresponding to the resonance $(1 : n)$, $n \geq 2$, can be analytically linearized if and only if it can be topologically linearized in the complex domain.*

Proof. Consider the trace of the foliation on the unit sphere. The first return map is a topological invariant of the foliation. For the nonlinear Jordan node (27.9) with $a \neq 0$ the holonomy map is nontrivial (its n th power has an isolated fixed point), whereas the holonomy map for the linear node is linear and its n th power identical. □

Note that in the real domain all nodes are topologically equivalent to each other.

Remark 27.16. The resonant conformal germ $\Delta \in \text{Diff}_1(\mathbb{C}, 0)$ has a fixed point at the origin and its n th iteration Δ^n is tangent to identity with order $n + 1$.

Hence the iterated power $\tilde{\Delta}^n$ of any sufficiently close conformal germ $\tilde{\Delta}$ will have $n + 1$ fixed points near the origin. One of these points is a fixed point for $\tilde{\Delta}$ by the implicit function theorem. The remaining n points form a tuple of n -periodic points that are positioned approximately at vertices of a regular n -gon and permuted by $\tilde{\Delta}$ cyclically.

In terms of the traces of the foliations, this means that a vector field obtained by a sufficiently small perturbation of the nonlinearizable resonant node, produces a foliation on \mathbb{S}_1^3 which has two cycles close to each other and linked with the index $n \geq 2$. All other leaves of the foliation are bi-asymptotic to these cycles. This gives the complete topological description for the *bifurcation of complex topological type* for passing through a Poincaré resonance. The assertion remains true also for the Jordan node (linear or not) with the ratio of eigenvalues equal to 1.

27D. Topological classification of linear complex flows in the Siegel domain. As opposite to the Poincaré case, the topological classification of holomorphic foliations generated by Siegel-type linear flows involves a number of continuous invariants. This means that in general an arbitrary small variation of the linear system results in a topologically different holomorphic foliation. This is a manifestation of the phenomenon known as *rigidity*.

Consider a hyperbolic linear vector field $\dot{x} = Ax$ of Siegel type in \mathbb{C}^n , i.e., assume that the origin belongs to the convex hull of the eigenvalues $\lambda_1, \dots, \lambda_n$ of A ; see §5A. The complex hyperbolicity in the sense of Definition 27.7 implies that the matrix A can be assumed diagonal, $A = \Lambda = \text{diag}\{\lambda_1, \dots, \lambda_n\}$, and the origin is necessarily *in the interior* of the convex hull $\text{conv}\{\lambda_1, \dots, \lambda_n\} \subseteq \mathbb{C}$. In particular, hyperbolic Siegel systems exist only when $n \geq 3$.

Hyperbolicity means that the invariant axes (diagonalizing coordinates) of the linear vector field can be ordered to meet the following condition:

$$\begin{aligned} \dot{x} &= \Lambda x, & x &\in \mathbb{C}^n, & \Lambda &= \text{diag}\{\lambda_1, \dots, \lambda_n\}, & n &\geq 3, \\ \text{Im } \lambda_{j+1}/\lambda_j &< 0, & j &= 1, \dots, n, & 0 &\in \text{conv}\{\lambda_1, \dots, \lambda_n\}. \end{aligned} \tag{27.10}$$

Here and everywhere below the enumeration of coordinates is cyclical modulo n , so that the assumption (27.10) includes the condition $\text{Im } \lambda_n/\lambda_1 < 0$ as well. Denote by $\Phi^t = \exp t\Lambda: \mathbb{C}^n \rightarrow \mathbb{C}^n$ the *complex* flow of the linear system $\dot{x} = \Lambda x$ and \mathcal{F} the (singular) holomorphic foliation by phase curves of this flow:

$$\mathcal{F} = \{L_x\}_{x \neq 0}, \quad L_x = \{\Phi^t(x) : t \in \mathbb{C}\}.$$