

If G is cyclic and generated by a holomorphism g , then we say that u is a *first integral* of g . The germ g is *integrable* if it admits a nontrivial holomorphic first integral.

Proposition 6.25. *An holomorphism is periodic if and only if it is integrable.*

More precisely, $h \in \text{Diff}(\mathbb{C}, 0)$ admits a first integral $u(z) = cz^m + \dots$, $c \neq 0$, if and only if $h^k = \text{id}$, where k divides m .

Proof. A periodic holomorphism h is linearizable by Theorem 6.7 and any linear map $x \mapsto \nu x$, $\nu^k = 1$, has the first integrals $u(z) = z^m$ for all m divisible by k (the case $m = 0$ is trivial and has to be excluded).

Conversely, if h is integrable and $u(z) = z^m + \dots$ is the integral, then every level set $M_c = \{u(z) = c\} \subseteq (\mathbb{C}, 0)$ in a sufficiently small neighborhood of 0 consists of exactly m points that are permuted by h . By the Lagrange theorem, $h|_{M_c}$ is of period $k = k(c)$ that divides m . Let k be the minimal value such that the set of k -periodic points is infinite. Then the k th iterate of h is identity by the uniqueness theorem. \square

From this proposition and Theorem 6.9 we immediately derive the following necessary condition of integrability.

Corollary 6.26. *An integrable group is finite cyclic (commutative).* \square

Remark 6.27. Any germ of a holomorphic function $u(z) = cz^m + \dots$ of finite order m admits a cyclic symmetry group of order m . The group is generated by the germ of a self-map f which is the linear rotation by the primitive root of unity of order m in the holomorphic chart $w = z \cdot (c + \dots)^{1/m}$, in which the function itself becomes a monomial.

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Thus far we concentrated on commutative (finite or infinite) and metabelian groups, which are relatively tame. As was already shown, they admit simple formal classification based on the formal type of a single nontrivial parabolic element from the group. Topological classification of solvable groups is also relatively simple and can be derived from Theorem 21.2 (see §21) which claims that the only topological invariant of a parabolic germ is its level. The analytic classification of solvable groups of germs can be reduced to that of the nontrivial parabolic element as above. The corresponding analytic theory is developed in §21 and involves *nonpolynomial normal forms*; see Chapter IV. In summary,

- (1) dynamics of solvable groups is relatively simple, in particular,
- (2) they have no limit cycles, and

- (3) their analytic classification is much finer than the formal one, and the latter in turn is finer than topological classification.

For nonsolvable groups all of these properties fail. In the remaining part of this section we will show that a *generic* (nonsolvable) finitely generated group:

- (1) has dense orbits, among which
 (2) there exist countably many (properly defined) complex limit cycles.
 Moreover,
 (3) generic groups are *rigid*: two such groups can be topologically equivalent if and only if they are analytically equivalent.

These phenomena will again manifest themselves for singular holomorphic foliations on \mathbb{P}^2 : the subject will be treated in detail in §28.

The term *generic* in application to finitely generated pseudogroups will mean the following. We fix the number n (usually 2 or more) of generating germs and say that a certain property is *generic*, if it holds for all n -tuples of germs whose jets of some finite order r belong to a “massive” (say, open dense or full measure) subset of the total jet space $\bigoplus_{n \text{ times}} J^r(\mathbb{C}, 0)$.

Example 6.28. A generic group with $n \geq 2$ generators is noncommutative and, moreover, nonsolvable.

Indeed, both generators generically are hyperbolic (their multipliers are off the unit circle). Since the above definition of genericity does not depend on the choice of the chart, without loss of generality, we may assume that one of the generators, f_1 , is linear hyperbolic. The group will be noncommutative if the second generator f_2 in this chart is nonlinear (the second Taylor coefficient of f_2 is nonzero).

The commutator $h = [f_1, f_2]$ will be a parabolic element which is generically of level 1 (i.e., tangent to identity with a quadratic nonlinearity). Another parabolic element of level 1 is the commutator $[f_1, h]$. One can show that generically $[[f_1, h], h]$ will be nonzero and hence, by (6.6), have level 2 or more, which would imply nonsolvability.

Usually we will omit routine checks that a certain collection of requirements is fulfilled for a generic finite generated group: in more details various properties determined by finite or infinite order jets, will be discussed in §10, where the notion of *decidable* properties will be introduced.

6D. Dynamics generated by finitely generated groups of germs: pseudogroups. We need first to introduce a proper language for describing *dynamical properties* of finitely generated groups of conformal germs.

If a group G acts (in an abstract manner) on a space X , then the *orbit* of a point $x \in X$ is defined as the subset $G(x) = \{g \cdot x : g \in G\} \subseteq X$. However, if the elements of the group are not defined on the whole space X , then the definition of an orbit requires appropriate modification.

This caveat is especially important when G is the holonomy group of a holomorphic foliation. By the very definition of holonomy, if a point $a \in \tau$ on the cross-section belongs to the domain of the holonomy map Δ_γ , then the points a and $b = \Delta_\gamma(a)$ belong to the same leaf of the foliation. Thus orbits of the holonomy group understood as images of all well-defined holonomy maps, describe intersection of leaves of the foliation with a fixed cross-section.

We introduce a relaxed notion of a *pseudogroup* which differs from a group by the fact that the composition is not always defined. For our purposes it is sufficient to define pseudogroups of holomorphic maps whose domains are open subsets of \mathbb{C} containing a common fixed point (the origin); the modification for the general case can be made following the same lines.

Definition 6.29. Let U be a neighborhood of the origin in \mathbb{C} and $G \subseteq \text{Diff}(\mathbb{C}, 0)$ an *arbitrary* subgroup of the group of germs. A *pseudogroup* Γ associated with G is a collection of pairs (f_α, U_α) , indexed by some index set $\alpha \in A$, such that $U_\alpha \subseteq U$ is an open set containing the origin, $f_\alpha: U_\alpha \rightarrow U$ is a holomorphic map defined (at least) in U_α and the group G consists of the germs at the origin of all maps f_α from the pseudogroup Γ .

Composition of two elements (f_α, U_α) and (f_β, U_β) is defined as the pair $(f_\alpha \circ f_\beta, U_{\alpha\beta})$ if and only if $U_{\alpha\beta} \subseteq U_\beta$ and $f_\beta(U_{\alpha\beta}) \subseteq U_\alpha$.

In other words, each conformal germ $\hat{f} \in G$ (in particular, the neutral element $\widehat{\text{id}} \in G$) is represented by many different maps f_α with different, in general, domains (of course, the maps coincide on the pairwise intersections of their domains).

The natural way to associate a pseudogroup Γ with any *finitely generated* group $G = \langle \hat{f}_1, \dots, \hat{f}_r \rangle \subset \text{Diff}(\mathbb{C}, 0)$ is as follows (we temporarily use the hats to distinguish between germs at the origin and holomorphic maps). Choose any collection of *representatives* $f_j^\pm: U_j \rightarrow \mathbb{C}$, $j = 0, \dots, r$, of the germs $\hat{f}_1^{\pm 1}, \dots, \hat{f}_r^{\pm 1}$ generating G . Then with an arbitrary word $w = (w_{j_n}^\pm w_{j_{n-1}}^\pm \dots w_{j_2}^\pm w_{j_1}^\pm) \in \mathfrak{F}_r$ (an element in the free group on r symbols, written from right to left) we can associate the conformal map f_w as the composition $f_{j_n}^\pm \circ f_{j_{n-1}}^\pm \circ \dots \circ f_{j_2}^\pm \circ f_{j_1}^\pm$ defined in the maximal domain U_w on which *all partial compositions*

$$f_{j_1} = f_{j_1}^\pm, \quad f_{j_2 j_1} = f_{j_2}^\pm \circ f_{j_1}^\pm, \quad \dots, \quad f_{j_n \dots j_1} = f_{j_n}^\pm \circ \dots \circ f_{j_2}^\pm \circ f_{j_1}^\pm$$

are well defined. Associating this domain (obviously, open and containing the origin) with the map $f_{j_n \dots j_1} = f_w$ representing the respective germ \widehat{f}_w , we obtain a pseudogroup. Choosing a different collection of the initial domains U_1, \dots, U_r formally results in a different pseudogroup, though most properties would not be affected.

If there are nontrivial identities in the group G , then the same germ admits several representatives with eventually different domains. To distinguish between such elements, we will remember together with each element $(f_\alpha, U_\alpha) \in \Gamma$ of the pseudogroup Γ the corresponding word w_α in the free group \mathfrak{F}_r . The corresponding collection of triples

$$\{(f_w, U_w, w) : w \in \mathfrak{F}_r, f_w \in \mathcal{O}(U_w)\} = \Lambda_G$$

will be called the *pseudogroup associated with the finitely generated group of conformal germs*. A triple (element of the pseudogroup) is *nontrivial*, if the corresponding word w is nontrivial in \mathfrak{F}_r , even if $f_w = \text{id}|_{U_w}$. Yet in most cases we will omit the third component to simplify the notation.

Remark 6.30. In order to avoid technical problems, we will always assume that if (f, U) belongs to a pseudogroup Γ , then all restrictions $(f|_V, V)$ for $V \subseteq U$, also belong to Γ .

For a pseudogroup Γ the notion of an orbit of a point can be introduced without any complications. A “periodic” orbit is naturally called a cycle.

Definition 6.31. The orbit of a point $x \in U$ by a pseudogroup Γ is the set $\Gamma(x)$ of all points $f_\alpha(x)$ for all elements $(f_\alpha, U_\alpha) \in \Gamma$ such that $x \in U_\alpha$.

Definition 6.32. The point $x \neq 0$ is called a *cycle*, if it is fixed by a nontrivial element (f_α, U_α) of the pseudogroup, i.e., $x \in U_\alpha$ and $f_\alpha(x) = x$ (thus for a cyclic group *all* points are cycles). The cycle is *limit* (in full, a complex limit cycle of a pseudogroup), if x is an *isolated* fixed point of f_α in U_α .

The notion of equivalence of groups of conformal germs translates naturally into equivalence of pseudogroups. Two pseudogroups Γ, Γ' are equivalent if there exists a conformal biholomorphism $h: (U, 0) \rightarrow (U', 0)$ such that Γ' consists of all pairs $(h \circ f_\alpha \circ h^{-1}, h(U_\alpha))$ such that $(f_\alpha, U_\alpha) \in \Gamma$ (subject to the technical convention from Remark 6.30). Clearly, equivalent pseudogroups have identical dynamical properties.

6E. Periodic orbits and periodic germs. To illustrate the usefulness of the notion of a pseudogroup, we establish simple dynamic properties of periodic (and aperiodic) germs.

Periodicity of a germ $\hat{g} \in \text{Diff}(\mathbb{C}, 0)$ (meaning that $\hat{g}^n = \text{id}$) implies that all g -orbits are periodic (cycles) for any representative g of \hat{g} . The inverse statement is less obvious.

Let $\hat{g} \in \text{Diff}(\mathbb{C}, 0)$ be a conformal germ that admits a representative g defined in an open set V containing the origin. For any set $U \subseteq V$ consider the restriction $g|_U$ and the ‘‘cyclical’’ pseudogroup Γ_U generated by the element (g, U) . For an arbitrary point $x \in U$ denote by $\Gamma(x|U)$ the Γ_U -orbit of the point x : by definition,

$$\Gamma(x|U) = \{g^n(x) : n \in \mathbb{Z}, \text{ and for all } k \text{ between } 0 \text{ and } n, g^k(x) \in U\}.$$

The orbit may be finite, in which case it consists of the consecutive iterates

$$g^{-n}(x), g^{-n+1}(x), \dots, g^{-1}(x), x, g(x), \dots, g^{m-1}(x), g^m(x)$$

for some $n, m \geq 0$, or infinite in one or both directions. We consider only *maximal* orbits, i.e., assume that $g^{-n-1}(x)$ and $g^{m+1}(x)$ already do not belong to U if n (resp., m) is finite. Note that the infinite orbit may consist of finitely many *distinct* points (if and only if the orbit is periodic).

Consider the integer-valued function $\nu(x)$ defined as the length of the maximal orbit,

$$\nu(x) = \nu(x|U) = \max\{m + n : g^{-n}(x), \dots, x, \dots, g^m(x) \in U\}. \quad (6.16)$$

If the orbit is infinite, we set $\nu(x) = +\infty$. By construction, *the function ν is constant along orbits of g .*

The continuity of g implies semicontinuity of the function ν : if U is open and $\nu(x) < +\infty$, then for all $y \in U$ sufficiently close to x , $\nu(y) \geq \nu(x)$. Conversely, if U is a closed subset of V and $\nu(x)$ is finite, then for all $y \in U$ sufficiently close to x , $\nu(y) \leq \nu(x)$. In the latter case if x is a point of discontinuity for ν , then the orbit $\Gamma(x|U)$ intersects the boundary ∂U . All these properties are immediate (Exercise 6.2).

Lemma 6.33. *If the germ $g \in \text{Diff}(\mathbb{C}, 0)$ is aperiodic, i.e., if the cyclic group $G = \{g^{\mathbb{Z}}\}$ is infinite, then for any small open domain $U \ni 0$ there are uncountably many infinite aperiodic orbits $\Gamma(x|U)$.*

Proof. Consider an arbitrary *closed* circular disk $D_\rho = \{|x| \leq \rho\}$ and its boundary circle $K_\rho = \partial D_\rho$, $\rho > 0$.

1. We prove that there are uncountably many points on D_ρ with infinite orbits in D_ρ . To that end, we will show that on each circle K_r , $r \leq \rho$, there is at least one point with an infinite orbit in $D_r \subseteq D_\rho$. Since the number of different circles which can intersect any given orbit is at most countable, this will prove that the number of infinite orbits is uncountable.

Assume that all points on the circumference K_r have finite orbits in D_r , i.e., the corresponding length function $\nu(\cdot) = \nu_{D_r}(\cdot)$ takes only finite values

on K_r . Since K_r is compact, this means that ν is bounded from above on K_r , and all orbits intersecting the boundary, are finite and no longer than some finite number $N \in \mathbb{N}$.

On the other hand, since $g(0) = 0$, the orbit of the origin $x = 0$ is infinite and $\nu(0) = +\infty$. Because of the semicontinuity of ν on the connected disk D_r , the function ν must have a discontinuity point $y \in D_r \setminus K_r$ with the value $\nu(y)$ strictly greater than N somewhere in the interior of D_r . Yet this means that the orbit $\Gamma(y|D_r)$ which is longer than N , intersects the boundary K_r . Since ν is constant along orbits, this contradicts the choice of N as the upper bound of ν on K_r .

2. To complete the proof of the lemma, note that the set of points with infinite orbits is the union of periodic points and the infinite aperiodic orbits. For each finite n , the n -periodic points inside D_r are roots of the equation $g^n(x) - x = 0$ and form a finite subset of D_r by the uniqueness theorem for the analytic germ g^n . The union of these finite sets is at most countable. Therefore the complement, the union of infinite aperiodic orbits in D_r , is uncountable. \square

Thus we have the following alternative.

Theorem 6.34. *Any finitely generated group $G \subset \text{Diff}(\mathbb{C}, 0)$ is either integrable, or any pseudogroup associated with G has uncountably many infinite aperiodic orbits.*

Proof. If G includes an aperiodic germ g , then this germ has uncountably many aperiodic orbits by Lemma 6.33. Conversely, if all elements of G are of finite order, then by Theorem 6.9 the group is finite and cyclic, hence linearizable. Its integrability follows from Proposition 6.25. \square

6F. Closure of a pseudogroup and density of orbits. Once a group of conformal germs is replaced by the pseudogroup, one can define the notions of convergence, closure, etc.

Definition 6.35. A sequence of elements $\{(f_j, U_j)\}_{j=1}^{\infty}$ of a pseudogroup Γ converges to an element $(g_*, U_*) \in \Gamma$, if $U_* \subseteq U_j$ for all j (starting from some number) and the restrictions $f_j|_{U_*}$ converge uniformly to g_* there.

The *closure* of a pseudogroup Γ is the collection of all limits of converging sequences of elements from Γ .

Clearly, the closure is again a pseudogroup, denoted by $\overline{\Gamma}$. The following statement is proved by the standard approximation arguments.

Proposition 6.36. *Let Γ be a pseudogroup of conformal maps and $\overline{\Gamma}$ its closure. If the orbit $\overline{\Gamma}(x)$ of some point is dense in an open domain U , then the orbit $\Gamma(x)$ of the initial pseudogroup is also dense there.* \square

This proposition is especially useful when the closure of a pseudogroup contains a sub-pseudogroup with dense orbits. This happens, as we will show, when the group of germs G contains a pair of hyperbolic germs with the multipliers generating a dense lattice of points in \mathbb{C} .

By Schröder–Koenigs Theorem 5.18, a hyperbolic germ is always linearizable: there exists a biholomorphism h conjugating g with the linear map $x \mapsto \mu x$. Replacing the pseudogroup Γ by an equivalent one, we may assume from the very beginning that Γ contains a linear hyperbolic map.

Consider again the multiplier homomorphisms $T: G \rightarrow \mathbb{C}^*$ and $T: \Gamma \rightarrow \mathbb{C}^*$ mapping each germ $f_\alpha \in G$ (resp., element $(f_\alpha, U_\alpha) \in \Gamma$) into its multiplier at the origin (defined independently of the choice of a chart). Denote the image of this application by Λ_G or Λ_Γ respectively: this is a multiplicative subgroup of \mathbb{C}^* .

Theorem 6.37. *If the pseudogroup $\Gamma = \{(f_\alpha, U_\alpha)\}$ contains a linear hyperbolic map $(\mu_0 x, U_0)$, $|\mu_0| \neq 1$, then the closure $\overline{\Gamma}$ contains also all linear maps $(\mu_\alpha x, \frac{1}{2}U_\alpha)$ for all $\mu_\alpha = df_\alpha(0)/dx \in \Lambda_\Gamma$.*

Proof. Since Γ is a pseudogroup associated with a group G which together with each germ contains its inverse, without loss of generality, we may assume that the multiplier μ_0 denoted for brevity by μ , is *contracting* as a linear map, $|\mu| < 1$.

Let $(g, V) \in \Gamma$ be an arbitrary element. By Remark 6.30 we may assume that V is a circular disk, so that $\mu V \subset V$. Hence all the elements (g_n, V) , $g_n = \mu^{-n} \circ g \circ \mu^n$, belong to Γ for all $n \geq 0$ (i.e., these compositions are all defined in V).

Expanding g into the Taylor series converging in V as $g(x) = \sum_{k=1}^{\infty} a_k x^k$, $a_1 = \lambda$, we conclude that the k th Taylor coefficient of g_n is $a_k \mu^{(k-1)n}$. As $n \rightarrow +\infty$, this tends to zero for all $k \geq 2$, which means that the elements g_n converge uniformly to the linear map $(\lambda x, V')$ for any $V' \Subset V$. \square

In the future we will often require the following condition imposed either on groups of conformal germs, or on the associated pseudogroups of maps.

Definition 6.38. A finitely generated group $G \subset \text{Diff}(\mathbb{C}^1, 0)$ (resp., a pseudogroup Γ) satisfies the *density condition* if the multiplicative subgroup Λ_G (resp., Λ_Γ) generated by multipliers of all germs (resp., maps) is dense in the multiplicative group \mathbb{C}^* :

$$\overline{\Lambda_G} = \mathbb{C} \supset \mathbb{C}^*, \quad \text{resp.,} \quad \overline{\Lambda_\Gamma} = \mathbb{C} \supset \mathbb{C}^*. \quad (6.17)$$

Example 6.39 (Closed subgroups of \mathbb{C}^*). Consider the exponential map $\exp 2\pi i: \mathbb{C} \rightarrow \mathbb{C}^*$, $\lambda \mapsto \exp(2\pi i\lambda)$, which is a topological nonramified covering, and finitely generated (multiplicative) subgroup $\subseteq \mathbb{C}^*$. The preimage of

G by this map is a *lattice of points* L , a \mathbb{Z} -module in $\mathbb{C} \cong \mathbb{R}^2$, which always contains the unity (and hence all the integers \mathbb{Z}): this lattice is generated by 1 and logarithms $\lambda_j = \frac{1}{2\pi i} \ln \mu_j$, $j = 1, \dots, n$, of the generators μ_1, \dots, μ_n of G . Obviously, G is dense in \mathbb{C}^* if and only if L is dense in \mathbb{C} .

The *closed* lattices of points in \mathbb{C} can be easily described: they can be only discrete (i.e., contain 0 as an isolated point) isomorphic to $\mathbb{Z} + \lambda_1 \mathbb{Z}$, the union of parallel translates of a line $\mathbb{Z} + \lambda_1 \mathbb{R}$ or $\mathbb{R} + \lambda_1 \mathbb{Z}$, and the whole plane \mathbb{C} . The latter case is generic if the number of generators is three or more (it is sufficient to demand that the three generators are nonresonant, i.e., $n_0 + n_1 \lambda_1 + n_2 \lambda_2$ never represents zero for a nontrivial choice of coefficients n_0, n_1, n_2). This implies that generically a multiplicative subgroup generated by two generators $\mu_1, \mu_2 \in \mathbb{C}^*$ is dense.

Other types of closed sublattices of points in \mathbb{C} produce the following closed subgroups of \mathbb{C}^* (for the sake of completeness we include \mathbb{C}^* as well):

- (1) \mathbb{C}^* (the entire group);
- (2) $\mathbb{Z}_p \times \mathbb{R}_+^*$, $1 \leq p < \infty$ (finite number of spirals that eventually degenerate into rays);
- (3) $2^{\mathbb{Z}} \times \mathbb{T}$ (infinite many circles with the radii forming a geometric progression);
- (4) $\mathbb{Z}_p \times 2^{\mathbb{Z}}$ (finite number of complex geometric progressions).

In this list $2^{\mathbb{Z}}$ stands for the infinite cyclic multiplicative subgroup of \mathbb{C}^* , $\mathbb{T} = \{|\mu| = 1\} \cong \mathbb{R}/\mathbb{Z}$ is the unit circle (considered as the a multiplicative group) and $\mathbb{Z}_p \subset \mathbb{T}$ is the group of roots of unity of degree p .

Example 6.39 explains why the density condition (6.17) is *generic*: for any $n \geq 2$ the tuple of germs (f_1, \dots, f_n) , $f_j \in \text{Diff}(\mathbb{C}, 0)$ such that the group generated by $f_j^{\pm 1}$ satisfies (6.17), form a dense subset in the space of all tuples $(\text{Diff}(\mathbb{C}, 0))^n$.

Corollary 6.40 (Density theorem for generic pseudogroups). *If a pseudo-group Γ satisfies the density condition (6.17), then for any $x \in U$ there exists a small neighborhood V such that the orbit $\Gamma(x)$ is dense in V .*

Proof of the corollary. If two multipliers generate a dense subgroup of \mathbb{C}^* , then both of them should be hyperbolic (off the unit circle), hence the closure of Γ contains two linear maps $x \mapsto \mu_j x$, $j = 1, 2$, defined in two disks U_1, U_2 .

By Proposition 6.36, to show density of orbits of the initial pseudogroup Γ it is sufficient to prove that the orbits of the pseudogroup generated by the two linear maps are dense.

Given any linear map $x \mapsto \mu x$, we can find a product $\prod \mu_{1,2}^{\pm 1}$ approximating μ with any degree of accuracy by (6.17). Using commutativity of \mathbb{C}^* , we can always rearrange the factors in this product so that all contracting terms come first and the expanding terms after. This would guarantee that the composition of the respective linear maps is well defined in the pseudogroup. \square

6G. Abundance of limit cycles for generic pseudogroups. Under the density assumptions one can show that a *noncommutative* pseudogroup has infinitely many distinct complex limit cycles accumulating to the origin.

Theorem 6.41. *A noncommutative finitely generated pseudogroup of conformal maps meeting the density condition, possesses infinitely many limit cycles accumulating to the origin.*

Both assumptions of the theorem (density condition and noncommutativity) are obviously generic.

Proof. Consider the maps of the pseudogroup Γ in the canonical chart linearizing one (hence all) hyperbolic germs belonging to it. Being noncommutative, Γ contains a nonidentical map (f, U) with multiplier 1. Rescaling the canonical chart, we may assume that U is large enough to contain the unit disk: $\mathbb{D} = \{|z| \leq 1\} \subseteq U$, $f(0) = 0$, $f(z) - z \neq 0$. The ratio $f(z)/z$ is a nonconstant holomorphic function that takes at least two distinct values, 1 (at the origin) and $\mu \neq 1$ at some other point a . Without loss of generality we may assume that $|a| < \frac{1}{3}$.

By Theorem 6.37 and the density condition (6.17), the closure of the pseudogroup contains the linear map $g(z) = \mu z$ defined on $\frac{1}{2}\mathbb{D}$, i.e., there exists an element $h \in \Gamma$ of the pseudogroup approximating g with arbitrary high accuracy on $\frac{1}{2}\mathbb{D}$. The function $f(z) - g(z)$ has at least two isolated roots ($z = 0$ and $z = a$) in $\frac{1}{2}\mathbb{D}$; by the argument principle, $f(z) - h(z)$ has at least the same number of roots in this domain if the approximation is accurate enough. In other words, the map $f^{-1} \circ h$ has at least two isolated fixed points in $\frac{1}{2}\mathbb{D}$, one at the origin, the other elsewhere. The latter point is the limit cycle.

Clearly, this construction can be repeated with U replaced by an arbitrarily small neighborhood of the origin. This shows that limit cycles accumulate to the origin, as asserted by the theorem. \square

Remark 6.42. Theoretically, all limit cycles constructed in the proof of Theorem 6.41, can belong to a single orbit of the pseudogroup.

6H. Rigidity of finitely generated groups of conformal germs. The term “rigidity” will repeatedly appear in this book in connection with various phenomena sharing a common feature that “weaker equivalence implies stronger equivalence”. Rigidity appears in the context where we consider objects for which there is a hierarchy of equivalence relations of various strength (topological, differentiable, holomorphic). An object is rigid when any weaker equivalence between it and any other object means that the two objects are strongly equivalent.

One rigidity-type result was already observed in Remark 4.6, when weaker (formal) equivalence automatically implies analytic equivalence.

In general, however, rigidity deals with an interplay between topological and stronger (smooth, analytic, *etc.*) classifications. In this more restricted context rigidity means that there is no way to change the weak structure (topology) in a nontrivial way (i.e., without changing all other, finer structures of the object).

Example 6.43. The sphere is rigid in the class of Riemann surfaces: any other Riemann surface topologically equivalent to the sphere, is conformally equivalent to it.

All the way around, the tori are not rigid: a two-dimensional torus \mathbb{T}^2 has conformal invariants.

The simplest rigidity-type property can be observed for finitely generated groups of conformal germs.

Definition 6.44. A finitely generated group of germs $G \subset \text{Diff}(\mathbb{C}, 0)$ is called *rigid*, or *topologically rigid*, if any germ of a homeomorphism h topologically conjugating G with another group $G' \subset \text{Diff}(\mathbb{C}, 0)$ is necessarily conformal, $h \in \text{Diff}(\mathbb{C}, 0)$.

Sufficient conditions for rigidity are the same as for Theorem 6.41 on abundance of limit cycles.

Theorem 6.45 (Rigidity theorem for groups of conformal germs). *A non-commutative finitely generated pseudogroup Γ of conformal maps meeting the density condition, is rigid.*

Moreover, if Γ'_t is a family of pseudogroups analytically depending on a complex parameter $t \in U \subset \mathbb{C}^p$ and topologically equivalent to the pseudogroup Γ with the above listed properties for all t , then there exists a holomorphic conjugacy $h_t: (\mathbb{C}^1, 0) \rightarrow (\mathbb{C}^1, 0)$ between Γ'_t and Γ which analytically depends on t .

We start the proof of Theorem 6.45 by analyzing topological conjugacies between dense subgroups of the commutative multiplicative group \mathbb{C}^* .

Proposition 6.46. *Let $G, G' \subseteq \mathbb{C}^*$ be two finitely generated dense subgroups topologically conjugated by a homeomorphism $h: (\mathbb{C}, 0) \rightarrow (\mathbb{C}, 0)$ preserving the orientation.*

Then $h(z) = cz|z|^\beta$ for some complex numbers $c \in \mathbb{C}^$ and $\beta \in \mathbb{C}$.*

Proof. The topological conjugacy between the groups means that there exists a multiplicative group isomorphism $A: G \rightarrow G' \subseteq \mathbb{C}^*$ and a homeomorphism $h: (\mathbb{C}, 0) \rightarrow (\mathbb{C}, 0)$ such that

$$h(\mu z) = A(\mu)h(z), \quad \forall z \in (\mathbb{C}, 0), \quad \forall \mu \in G. \quad (6.18)$$

1. We claim first that $A(\mu) = \mu|\mu|^\beta$ for some complex value of $\beta \in \mathbb{C}$.

The automorphism A satisfies the multiplicativity condition

$$A(\mu\nu) = A(\mu)A(\nu), \quad \forall \mu, \nu \in G, \text{ hence } A(1) = 1, \quad (6.19)$$

and the functional equation (6.18) implies immediately that both A and its inverse are *continuous* as complex-valued functions of μ . Thus without loss of generality we may assume that A (resp., A^{-1}) is defined on the closed subgroups \overline{G} , resp., $\overline{G'}$, i.e., $A: \mathbb{C}^* \rightarrow \mathbb{C}^*$ is a homeomorphism.

2. Intuitively, the functional equation (6.19) becomes additive after passing to logarithms. Yet since the logarithm is multivalued, one has to exercise some extra care. In the special case where $\overline{G} = \mathbb{C}^*$, one can choose the *continuous* branch of logarithm. More precisely, there exists a continuous complex function $\widehat{A}: \mathbb{C} \rightarrow \mathbb{C}$ which covers the automorphism $A: \mathbb{C}^* \rightarrow \mathbb{C}^*$ by the exponential map:

$$A(\exp 2\pi iw) = \exp 2\pi i\widehat{A}(w), \quad w \in \mathbb{C}. \quad (6.20)$$

If normalized by the condition $\widehat{A}(0) = 0$, it becomes uniquely defined and the multiplicative identity (6.19) implies that \widehat{A} is additive modulo \mathbb{Z} ,

$$\widehat{A}(\lambda + \lambda') = \widehat{A}(\lambda) + \widehat{A}(\lambda') \pmod{\mathbb{Z}} \quad \forall \lambda, \lambda' \in \mathbb{C}.$$

The integer number $N(\lambda, \lambda') = \widehat{A}(\lambda + \lambda') - \widehat{A}(\lambda) - \widehat{A}(\lambda')$ is a continuous, hence constant function of $(\lambda, \lambda') \in \mathbb{C}^2$: because of the normalizing condition $\widehat{A}(0) = 0$ we have $N(0, 0) = 0$ and therefore the application \widehat{A} is truly additive,

$$\widehat{A}(\lambda + \lambda') = \widehat{A}(\lambda) + \widehat{A}(\lambda'), \quad \forall \lambda, \lambda' \in \mathbb{C}. \quad (6.21)$$

This additivity implies that $m \cdot \widehat{A}(\frac{1}{m}\lambda) = \widehat{A}(\lambda)$ for any natural m ; since \widehat{A} is one-to-one, we conclude that $\widehat{A}(\frac{1}{m}\lambda) = \frac{1}{m}\widehat{A}(\lambda)$ and therefore \widehat{A} is \mathbb{Q} -linear map of \mathbb{C} into itself. Finally, the continuity of \widehat{A} means that \widehat{A} is an \mathbb{R} -linear automorphism of $\mathbb{C} \cong \mathbb{R}^2$. Any such automorphism necessarily has the form

$$\widehat{A}(\lambda) = a\lambda + b\bar{\lambda}, \quad \text{for some } a, b \in \mathbb{C}, \quad |a| \neq |b|. \quad (6.22)$$

The orientation is preserved if $|a| > |b|$ and reverted otherwise.

The \mathbb{R} -linear map \widehat{A} covers the multiplicative map A by the logarithmic covering $\lambda \mapsto \mu = \exp 2\pi i \lambda$. Therefore $\widehat{A}(1)$ must be an integer number $n \in \mathbb{Z}$. This means that

$$\widehat{A}(\lambda) = n\lambda + \frac{1}{2}\beta(\lambda - \bar{\lambda}), \quad (6.23)$$

for some $\beta \in \mathbb{C}$, and hence

$$\begin{aligned} A(\mu) &= \exp 2\pi i \left\{ n \cdot \frac{\ln \mu}{2\pi i} + \frac{\beta}{2} \left[\frac{\ln \mu}{2\pi i} - \overline{\left(\frac{\ln \mu}{2\pi i} \right)} \right] \right\} \\ &= \mu^n \exp \frac{\beta}{2} [\ln \mu + \ln \bar{\mu}] = \mu^n |\mu|^\beta. \end{aligned}$$

It remains to notice that by (6.18) the map A must be an orientation-preserving homeomorphism, which leaves only one possibility, $n = 1$. This proves the equality $A(\mu) = \mu |\mu|^\beta$.

3. If the homeomorphism h solving the functional equation (6.18) is represented under the form $h(z) = z|z|^\beta f(z)$ with the same β as before and some complex-valued function f continuous on $(\mathbb{C}^*, 0)$, then from the functional equation (6.18) we obtain after cancellation of all terms the trivial “functional equation” on f ,

$$f(\mu z) = f(z), \quad \forall \mu \in G, z \in (\mathbb{C}, 0).$$

Since G is dense in \mathbb{C}^* , we conclude that $f(z) \equiv c \neq 0$ must be a constant. This completes the proof of the proposition. \square

Remark 6.47. Passing to a different chart in the preimage or the image, one can always assume that $c = 1$.

Remark 6.48. If h is a homeomorphism *reverting* the orientation and topologically conjugating G with G' as in (6.18), then $A(\mu) = \bar{\mu} |\mu|^\beta$ and $h(z) = c\bar{z}|z|^\beta$. This corresponds to the case $n = -1$ in (6.23). To prove that, it is sufficient to replace $h(z)$ by $\bar{h}(z) = \overline{h(\bar{z})}$ which is an orientation-preserving homeomorphism conjugating the two groups $\bar{G}, \bar{G}' \subseteq \mathbb{C}^*$ obtained from G and G' by conjugation with the mirror symmetry $z \mapsto \bar{z}$.

Remark 6.49. From the proof of Proposition 6.46 it follows that two dense multiplicative subgroups $\langle \mu_1, \dots, \mu_n \rangle$ and $\langle \mu'_1, \dots, \mu'_n \rangle$ are topologically conjugate if and only if there exists an \mathbb{R} -linear map $\widehat{A}: \mathbb{C} \rightarrow \mathbb{C}$ which takes 1 to 1 and establishes a one-to-one correspondence between logarithms of the generators modulo integers for *some* choice of the branch of logarithm:

$$\widehat{A}\lambda_j = \lambda'_j \bmod \mathbb{Z}, \quad \lambda_j = \frac{\ln \mu_j}{2\pi i} \bmod \mathbb{Z}, \quad \lambda'_j = \frac{\ln \mu'_j}{2\pi i} \bmod \mathbb{Z}. \quad (6.24)$$

From this observation and the topological invariance of the holonomy at infinity we can already conclude that the topological classification of certain classes of foliations is nondiscrete.

Now we can prove the main result of this subsection, the Rigidity theorem for finitely generated groups of conformal germs.

Proof of Theorem 6.45. Let $G = \langle f_1, \dots, f_n \rangle$ and $G' = \langle f'_1, \dots, f'_n \rangle$ be two topologically conjugated noncommutative groups of germs with G satisfying the density condition 6.17. Without loss of generality we may assume that f_1 is hyperbolic, and the corresponding multiplier μ_1 has modulus less than 1.

1. Consider the germ $f'_1 \in G'$ conjugated by h with f_1 . This germ is also hyperbolic, moreover, we can easily see that $|\mu'_1| < 1$. Indeed, choose representatives of f_1, f'_1 defined in small topological disks U and $U' = h(U)$ respectively, so small that $f_1(U) \Subset U$. Then $f'_1(U') \Subset U'$ and by the Schwarz lemma, $|\mu'_1| < 1$.

2. If a homeomorphism h conjugates G with G' , then a representative of h conjugates (topologically) the corresponding pseudogroups Γ and Γ' and also the respective closures $\overline{\Gamma}$ and $\overline{\Gamma}'$. By Theorem 6.37, each closure contains a dense subgroup of the multiplicative group \mathbb{C}^* . Thus a representative of h topologically conjugates two dense subgroups as in Proposition 6.46.

3. Applying Proposition 6.46, we obtain an explicit description of the conjugating homeomorphism h : there exist holomorphic charts on U and U' (linearizing the hyperbolic germs f_1, f'_1 respectively) in which $h(z) = z|z|^\beta$.

4. If the commutator $[G, G']$ is nontrivial, it must contain a parabolic germ $f(z) = z + az^{n+1} + \dots \in \text{Diff}_1(\mathbb{C}, 0)$ which is conjugated by a homeomorphism $h(z) = z|z|^\beta$ with another parabolic element $f'(z') = z' + a'z'^{n'+1} + \dots \in [G', G']$. Clearly, $n = n'$ since this number is a topological invariant of germs (related to the number of petals; see §21). We will show that $\beta = 0$ so that $h(z) = z$.

To see this, we substitute the explicit form $h(z) = z|z|^\beta$ found in Proposition 6.46 into the conjugacy equation $h \circ f = f' \circ h$. After substitution, division by $z|z|^\beta$ and subtraction of 1 from each side we obtain the equation

$$z^n [a + \frac{\beta}{2}(a + \bar{a})] + \dots = a' z^n |z|^{n\beta} + \dots$$

where the dots stay for the terms decreasing as $|z| \rightarrow 0$ faster than the terms explicitly written on each side. Note that the principal term of left hand side after restriction on each circle $|z| = r$ is a trigonometric polynomial of degree n , whereas the principal part of the right hand side is a trigonometric polynomial of the same degree n only if $\beta = 0$. Thus $h(z) = z$ is a linear, hence holomorphic, map.

5. To prove that the conjugacy h analytically depends on additional parameters t , we proceed as follows. The condition that $h(z) = cz$ conjugates any generator $f_{j,t}$ with $f'_{j,t}$ translates into an infinite number of analytic

conditions on c and t . Thus the entire set $Q = \{(t, c) : f_{j,t}(cz) = cf'_{j,t}(z), j = 1, \dots, r\}$ is analytic near the point $(0, 1)$. If $f_t(z) = z + a_t z^{n+1} + \dots$ is conjugate with $f'_t(z) = z + a'_t z^{n+1} + \dots$, and $a_0 a'_0 \neq 0$, then $Q \subseteq \{c^n = a'_t/a_t\}$ (equating the coefficients before z^{n+1}). The latter analytic set consists of n analytic branches $c = c_k(t)$, $k = 1, \dots, n$. Since these branches are locally irreducible and $Q \cap \{t = \text{const}\}$ is nonempty for all t , the set Q contains at least one such branch. This branch gives the holomorphic dependence of $h(z) = c(t)z$ on t .

The proof of Theorem 6.45 is complete. \square

6I. Relaxing the genericity assumptions. Though the assumptions of noncommutativity and density required in Theorems 6.41 and 6.45 are generic, they fail for some important classes of finitely generated groups. For instance, the density condition fails for groups with resonant multipliers; such groups constitute a dense subspace in $\text{Diff}(\mathbb{C}^1, 0)$.

Yet the assumptions of the above theorems can be relaxed to an *open and dense* condition of *nonsolvability*; see Example 6.28. We formulate here without proofs several results in this direction.

Theorem 6.50 (A. Shcherbakov [Shc84], I. Nakai [Nak94]). *A nonsolvable finitely generated group $G \subset \text{Diff}(\mathbb{C}^1, 0)$ is rigid.*

The following result suggests that certain rigidity-like properties may occur even in the infinite cyclic subgroups. Recall that a germ is called elliptic, if its multiplier μ has modulus one: $\mu = \exp 2\pi i\varphi$, $\varphi \in \mathbb{R}/\mathbb{Z}$.

Theorem 6.51 (V. A. Naishul [Naï82]; see also [GLCP96]). *Suppose that two elliptic germs of conformal maps $f, f' \in \text{Diff}(\mathbb{C}, 0)$ are topologically conjugate by an orientation-preserving homeomorphism. Then the multipliers of f and f' coincide.*

This theorem is relatively easy in the case where the multipliers are roots of unity or Diophantine irrationalities (cf. with §5E). The real difficulties occur in the Cremer case.

Nonsolvability turns out also sufficient for existence of infinitely many limit cycles.

Theorem 6.52 (A. Shcherbakov [Shc86]; see also [BLL97] and [SRO98]). *A nonsolvable finitely generated group $G \subset \text{Diff}(\mathbb{C}^1, 0)$ possesses infinitely many complex limit cycles accumulating to the origin.*

The density of orbits obviously fails under the single nonsolvability assumption. For instance, if a group $G \subset \text{Diff}(\mathbb{C}^1, 0)$ consists of real (i.e., preserving \mathbb{R}) germs with positive multipliers, then any orbit starting in the upper (resp., lower) open half-plane, remains in the same half-plane forever and hence cannot be dense. Yet this is the only possible deviation from the density pattern.

Theorem 6.53 (I. Nakai [Nak94], a weaker result was proved in [Shc82]). *If G is a nonsolvable subgroup of $\text{Diff}(\mathbb{C}^1, 0)$, then there exists the germ of a real analytic curve $K \subsetneq (\mathbb{C}^1, 0) \cong (\mathbb{R}^2, 0)$ invariant by G , such that orbits of any pseudogroup Γ associated with G are dense in the connected components (sectors) of $(\mathbb{C}^1, 0) \setminus K$.*

Exercises and Problems for §6.

Exercise 6.1. Prove the rigidity assertions from Example 6.43.

Exercise 6.2. Prove the properties of the length function ν introduced in §6E.

Problem 6.3. Compute all holonomy maps of an integrable foliation $\{du = 0\}$, $u \in \mathcal{O}(\mathbb{C}^2, 0)$, if $u = \prod u_j^{p_j}$ is the primary decomposition of the holomorphic germ u with irreducible factors u_j and natural exponents $p_j \in \mathbb{N}$.

Problem 6.4. Prove that a formally integrable holomorphic self-map (or a finitely generated group G of holomorphic germs of self-maps from $\text{Diff}(\mathbb{C}, 0)$) is also analytically integrable; cf. with Theorem 6.8.

Suggestion. Use the formal chart in which $\widehat{u}(z) = z^m$.

Problem 6.5. Prove that an (orbital) symmetry of a holomorphic vector field on $(\mathbb{C}, 0)$ is necessary holomorphic itself.

Problem 6.6. Construct a finitely generated subgroup $G \subset \text{Diff}(\mathbb{C}, 0)$, whose orbits are dense in each of the two half-planes $\{\pm \text{Im } z > 0\}$ separately, yet both half-planes are invariant by G .

Generalize this example and find a group whose orbits are dense in each of $2p$ invariant sectors in $(\mathbb{C}, 0)$ for any $p > 1$ (cf. with Theorem 6.53).

Problem 6.7 (formal rigidity of generic groups). Assume that two finitely generated subgroups $G, G' \subseteq \text{Diff}(\mathbb{C}, 0)$ are *formally* equivalent and one of these groups contains a hyperbolic germ. Prove that in such case G and G' are holomorphically equivalent, moreover, any formal conjugacy between them is necessarily holomorphic (convergent).

7. Holomorphic invariant manifolds

In this short section we show that under rather weak conditions one can eliminate enough nonresonant terms to ensure existence of *holomorphic invariant (sub)manifolds*. Recall that a holomorphic submanifold $W \subset (\mathbb{C}^n, 0)$ is invariant for a holomorphic vector field F , if the vector $F(x)$ is tangent to W at any point $x \in W$. Traditionally the prefix ‘sub’ is omitted, though it plays an important role: in §14 we will discuss invariant analytic subvarieties that are *not* submanifolds because of their singularity.

7A. Invariant manifolds of hyperbolic singularities. Suppose that the spectrum $S \subset \mathbb{C}$ of linearization matrix A of a holomorphic vector field consists of two parts $S^\pm \subset \mathbb{C}$ separated by a real line (i.e., each part belongs to an open half-plane bounded by the line). In this case no eigenvalue from one part can be equal to a linear combination of eigenvalues from the other part with nonnegative coefficients,

$$\begin{aligned} \lambda_j^- - \sum \alpha_i \lambda_i^+ &\neq 0, & \lambda_i^+ - \sum \alpha_j \lambda_j^- &\neq 0, \\ \lambda_i^+ \in S^+, & \lambda_j^- \in S^-, & \alpha_i, \alpha_j \in \mathbb{Z}_+, & \end{aligned} \quad (7.1)$$

(we say that there are no *cross-resonances* between the two parts). Without loss of generality A can be assumed to be in the block diagonal form. By