3. Formal flows and embedding theorem

The assumption on convergence of Taylor series for the right hand sides of differential equations and their respective solutions is a very serious restriction: if it holds, then one can use various geometric tools as described in §2. However, considerable information can be gained without the convergence assumption, on the level of *formal power* (*Taylor*) *series*. For natural reasons, the corresponding results have more algebraic flavor.

In this section we introduce the class of formal vector fields and formal morphisms (self-maps), and prove that the flow of any such formal field can be correctly defined as a formal automorphism. The correspondence "field $\mapsto$ flow" can be inverted for maps with unipotent linearization: as was shown by F. Takens in 1974, any such formal map can be embedded in a unique formal flow [**Tak01**]. In §4 we establish classification of formal vector fields by the natural action of formal changes of variables.

3A. Formal vector fields and formal self-maps. For convenience, we will always assume that all Taylor series are centered at the origin, and use the standard multi-index notation: for $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{Z}_+^n$ we denote $|\alpha| = \alpha_1 + \cdots + \alpha_n$ and $\alpha! = \alpha_1! \cdots \alpha_n!$.

Definition 3.1. A *formal* (Taylor) series at the origin in $\mathbb{C}^n$ is the expression

$$f = \sum_\alpha c_\alpha x^\alpha, \qquad \alpha \in \mathbb{Z}_+^n, \quad c_\alpha \in \mathbb{C}. \tag{3.1}$$

The minimal degree $|\alpha|$ corresponding to a nonzero coefficient c_α, is called the *order* of f.

The set of all formal series is denoted by $\mathbb{C}[[x]] = \mathbb{C}[[x_1, \dots, x_n]]$. It is a commutative *infinite-dimensional* algebra over $\mathbb{C}$ which contains as a proper subset the algebra of germs of holomorphic functions, isomorphic to the algebra $\mathbb{C}\{x_1, \dots, x_n\}$ of *converging* series.

Definition 3.2. The *canonical basis* of $\mathbb{C}[[x]]$ is the collection of all monomials x^α, $\alpha \in \mathbb{Z}_+^n$, ordered in the following way: (i) all monomials of lower degree $|\alpha|$ precede all monomials of higher degree, and (ii) all monomials of the same degree are ordered lexicographically. This order will be denoted `deglex`-order.

Since the series may diverge, evaluation of $f(x_0)$ at any point $x_0 \in \mathbb{C}^n$ other than $x_0 = 0$, makes no sense. However, without risk of confusion we will denote the free term of a series $f \in \mathbb{C}[[x]]$ by $f(0)$ and the coefficient c_α by $\frac{1}{\alpha!}\frac{\partial^\alpha f}{\partial x^\alpha}(0)$. Under these agreements the Taylor formula becomes a *definition* of the Taylor series $f = \sum_{\alpha \geqslant 0} \frac{1}{\alpha!}\frac{\partial^\alpha f}{\partial x^\alpha}(0)\, x^\alpha$. Sometimes we write

$f(x)$ as an indication of the formal variables $x = (x_1, \dots, x_n)$ in which the series f depends.

All formal partial derivatives $\partial^\alpha f/\partial x^\alpha$ of a formal series f are well defined in the class $\mathbb{C}[[x]]$ as termwise derivatives.

The subset of $\mathbb{C}[[x]]$ which consists of formal series without the free term, is (as one can easily verify) a maximal ideal of the commutative ring $\mathbb{C}[[x]]$; it will be denoted by

$$\mathfrak{m} = \{f \in \mathbb{C}[[x]] \colon f(0) = 0\} = \Bigl\{ \sum_{|\alpha| \geqslant 1} c_\alpha x^\alpha \Bigr\}.$$

The maximal ideal is *unique* (again a simple exercise). In other words, the ring $\mathbb{C}[[x]]$ is a *local ring*.

For any finite $k \in \mathbb{N}$ the space of kth order jets can be described as the quotient

$$J^k(\mathbb{C}^n, 0) = \mathbb{C}[[x_1, \dots, x_n]]/\mathfrak{m}^{k+1}.$$

As a quotient ring, the affine finite-dimensional $\mathbb{C}$-space $J^k(\mathbb{C}^n, 0)$ inherits the structure of a commutative $\mathbb{C}$-algebra.

Definition 3.3. The *truncation* of formal series to a finite order k is the canonical projection map $j^k \colon \mathbb{C}[[x]] \to J^k(\mathbb{C}^n, 0)$, $f \mapsto f \bmod \mathfrak{m}^{k+1}$.

The name comes from the natural identification of $J^k(\mathbb{C}^n, 0)$ with polynomials of degree $\leqslant k$ in the variables $x_1, \dots, x_n$. If $l > k$ is a higher order, then $\mathfrak{m}^{l+1} \subset \mathfrak{m}^{k+1}$ so that the truncation operator j^k naturally "descends" as the projection $J^l(\mathbb{C}^n, 0) \to J^k(\mathbb{C}^n, 0)$ which will also be denoted by j^k.

In other words, a formal Taylor series $f \in \mathbb{C}[[x]]$ uniquely defines the k-jet $j^k f$ of any finite order k so that $\mathbb{C}[[x_1, \dots, x_n]]$ is in a sense the limit of the jet spaces $J^k(\mathbb{C}^n, 0)$ as $k \to \infty$. We will sometimes refer to formal series as *infinite* jets and write $\mathbb{C}[[x_1, \dots, x_n]] = J^\infty(\mathbb{C}^n, 0)$.

The canonical monomial basis in $\mathbb{C}[[x]]$ projects into canonically `deglex`-ordered monomial bases in all jet spaces $J^k(\mathbb{C}^n, 0)$. Convergence in $\mathbb{C}[[x]]$ is defined via finite truncations.

Definition 3.4. A sequence $\{f_j\}_{j=1}^\infty \subset \mathbb{C}[[x]]$ is said to be convergent, if and only if all its truncations $j^k f_j$ converge in the respective finite-dimensional k-jet space $J^k(\mathbb{C}^n, 0)$ for any finite $k \geqslant 0$.

Remark 3.5 (important)**.** All formal algebraic constructions described above can be implemented over the field $\mathbb{R}$ rather than $\mathbb{C}$ as the ground field. Moreover, for future purposes we will need the algebra $\mathfrak{A}[[x]]$ of formal power series in the indeterminates $x = (x_1, \dots, x_n)$ with the coefficients belonging to more general $\mathbb{C}$- or $\mathbb{R}$-algebras $\mathfrak{A}$. The principal examples are the algebras $\mathfrak{A} = \mathbb{C}[\lambda_1, \dots, \lambda_m]$ of polynomials in auxiliary indeterminates or the algebra $\mathcal{A} = \mathcal{O}(U)$ of holomorphic functions of additional variables $\lambda_1, \dots, \lambda_m$.

After introducing the algebra of "formal functions" we can define formal vector fields and formal maps via their algebraic (functorial) properties as in §1**G**.

With any *vector formal series* $F = (F_1, \dots, F_n)$ (n-tuple of elements from $\mathbb{C}[[x]]$) one can associate a derivation $\mathbf{F} = \sum_1^n F_j \partial/\partial x_j \in \operatorname{Der} \mathbb{C}[[x]]$ of the algebra $\mathbb{C}[[x]]$, a $\mathbb{C}$-linear application satisfying the Leibnitz rule (cf. with (1.27)),

$$\mathbf{F}\colon \mathbb{C}[[x]] \to \mathbb{C}[[x]], \qquad \mathbf{F}(gh) = g\,(\mathbf{F}h) + h\,(\mathbf{F}g).$$

Conversely, any derivation $\mathbf{F} \in \operatorname{Der} \mathbb{C}[[x]]$ is of the form $\mathbf{F} = \sum_1^n F_j \partial/\partial x_j$ with the components $F_j = \mathbf{F}x_j$. By *formal vector fields*, we mean both realizations, $F \in \mathbb{C}[[x]]^n$ or $\mathbf{F} \in \operatorname{Der} \mathbb{C}[[x]]$. The field F is said to have *singularity* (at the origin), if all these series are without free terms, $F_j(0) = 0$, $j = 1, \dots, n$.

The collection of formal vector fields will be denoted $\mathcal{D}[[\mathbb{C}^n, 0]]$. It is a $\mathbb{C}$-linear (infinite dimensional) space which possesses additional algebraic structures of the *module* over the ring $\mathbb{C}[[x]]$. The *commutator* (Lie bracket) of formal fields is defined in the usual way as $[\mathbf{F}, \mathbf{G}] = \mathbf{FG} - \mathbf{GF}$.

In a parallel way, a vector formal series $H = (h_1, \dots, h_n) \in \mathbb{C}[[x]]^n$ can be identified with an *automorphism* $\mathbf{H} \in \operatorname{Aut} \mathbb{C}[[x]]$ of the algebra $\mathbb{C}[[x]]$ if $H(0) = 0$, i.e., $h_j \in \mathfrak{m}$. Under this assumption, for any formal series $f = \sum_\alpha c_\alpha x^\alpha \in \mathbb{C}[[x]]$ one can correctly define the *substitution*

$$\mathbf{H}f(x) = f(H(x)) = \sum_{\alpha \geqslant 0} c_\alpha h^\alpha = \sum_{\alpha \geqslant 0} c_\alpha h_1^{\alpha_1}(x) \cdots h_n^{\alpha_n}(x). \tag{3.2}$$

Indeed, any k-truncation of $f(H(x))$ is completely determined by the k-truncations of f and H. We will say that $\mathbf{H}$ is *tangent to identity*, if $j^1\mathbf{H} = \mathrm{id}$.

The operator $\mathbf{H}$ defined by (3.2), is an *automorphism* of the algebra $\mathbb{C}[[x]]$, a $\mathbb{C}$-linear map respecting the multiplication,

$$\mathbf{H}\colon \mathbb{C}[[x]] \to \mathbb{C}[[x]], \qquad \mathbf{H}(fg) = \mathbf{H}f \cdot \mathbf{H}f.$$

Conversely, any homomorphism preserving convergence in $\mathbb{C}[[x]]$ is of the form $f \mapsto f \circ H$ for an appropriate vector series $H \in \mathbb{C}[[x]]^n$ with the *components* $h_j = \mathbf{H}x_j \in \mathbb{C}[[x]]$. By a *formal map* we mean either H or $\mathbf{H}$, depending on the context. If $\mathbf{H}$ is an homomorphism, then it must map the maximal ideal $\mathfrak{m} \subset \mathbb{C}[[x]]$ into itself and hence $h_j(0) = 0$, $j = 1, \dots, n$, which can be abbreviated to $H(0) = 0$.

If $\mathbf{H}$ is invertible (an isomorphism of the algebra $\mathbb{C}[[x]]$), we say it is a *formal isomorphism* of $\mathbb{C}^n$ at the origin. The collection of such isomorphisms forms a *group* denoted by $\operatorname{Diff}[[\mathbb{C}^n,0]]$ with the operation of composition. The latter can be defined either via substitution of the series, or as the composition of the operators acting on $\mathbb{C}[[x]]$.

Since the maximal ideal $\mathfrak{m}$ is preserved by any formal map $\mathbf{H} \in \operatorname{Diff}[[\mathbb{C}^n,0]]$ and any *singular* formal vector field $\mathbf{F} \in \mathcal{D}[[\mathbb{C}^n,0]]$, $F(0)=0$,

$$\mathbf{H}(\mathfrak{m}) = \mathfrak{m}, \qquad \mathbf{F}(\mathfrak{m}) \subseteq \mathfrak{m},$$

truncation of the series at the level of k-jets commutes with the action of $\mathbf{H}$ and $\mathbf{F}$, therefore defining correctly the isomorphism $j^k\mathbf{H}\colon J^k(\mathbb{C}^n,0) \to J^k(\mathbb{C}^n,0)$ and derivation $j^k\mathbf{F}\colon J^k(\mathbb{C}^n,0) \to J^k(\mathbb{C}^n,0)$ respectively, which can be identified with the k-jets of the formal map H and the formal vector field F. We wish to stress that $j^k\mathbf{F}$ is defined as an automorphism of the finite-dimensional jet space only if $F(0)=0$.

3B. Inverse function theorem. For future purposes we will need the formal inverse function theorem.

Theorem 3.6. *Let H be a formal map with the linearization matrix $A = \left(\frac{\partial H}{\partial x}\right)(0)$ which is nondegenerate. Then H is invertible in* $\operatorname{Diff}[[\mathbb{C}^n,0]]$.

If $A = E$ is the identity matrix and $H = (h_1,\dots,h_n)$, $h_i(x) = x_i + v_i(x) \bmod \mathfrak{m}^{k+1}$, where v_i are homogeneous polynomials of degree $k \geqslant 2$, then the formal inverse map $H^{-1} = (h_1',\dots,h_n')$ has the components $h_i'(x) = x_i - v_i(x) \bmod \mathfrak{m}^{k+1}$.

Clearly, the first assertion of the theorem follows from the second assertion applied to the formal map $A^{-1}H$.

Recall that a finite-dimensional linear operator $A\colon \mathbb{C}^n \to \mathbb{C}^n$ is *unipotent*, if $A-E$ is nilpotent, $(A-E)^n = 0$.

Lemma 3.7. *If $H \in \operatorname{Diff}[[\mathbb{C}^n,0]]$ is a formal map with the identical linearization matrix $(\frac{\partial H}{\partial x})$, then its truncation $j^k\mathbf{H}$ considered as an automorphism of the finite-dimensional jet algebras $J^k(\mathbb{C}^n,0)$, is a unipotent map for any finite order k.*

Proof. For any monomial x^α from the canonical basis, $\mathbf{H}x^\alpha = x^\alpha+$(higher order terms)$=$ $x^\alpha+$(linear combination of monomials of higher `deglex`-order). ☐

Proof of Theorem 3.6. Consider the homomorphism $\mathbf{H} \in \operatorname{Aut}\mathbb{C}[[x]]$ and denote $\mathbf{N} = \mathbf{H} - \mathbf{E}$ the formal "finite difference" operator ($\mathbf{E} = \mathrm{id}$ denotes the identical operator), $\mathbf{N}f = f \circ H - f$ (in the sense of the substitution of formal series). By Lemma 3.7, all finite truncations $j^k\mathbf{N}$ are nilpotent.

Define the operator H^{-1} as the series

$$\mathbf{H}^{-1} = \mathbf{E} - \mathbf{N} + \mathbf{N}^2 - \mathbf{N}^3 \pm \cdots . \tag{3.3}$$

This series converges (in fact, stabilizes) after truncation to any finite order because of the above nilpotency, hence by definition converges to an operator on $\mathbb{C}[[x]]$ satisfying the identities $\mathbf{H}\circ\mathbf{H}^{-1}=\mathbf{H}^{-1}\circ\mathbf{H}=\mathbf{E}$. It is an homomorphism of algebra(s), since for any $a,b\in\mathbb{C}[[x]]$ and their images $a'=\mathbf{H}a$, $b'=\mathbf{H}b$ which also can be chosen arbitrarily, we have $\mathbf{H}(ab)=a'b'$ and therefore

$$\mathbf{H}^{-1}(a'b')=\mathbf{H}^{-1}\mathbf{H}(ab)=ab=(\mathbf{H}^{-1}a')(\mathbf{H}^{-1}b').$$

Direct computation of the components of the inverse map yields

$$h_i'=\mathbf{H}^{-1}x_i=x_i-\mathbf{N}x_i+\cdots=x_i-(h_i(x)-x_i)+\cdots=x_i-v_i(x)+\cdots$$

as asserted by the theorem. □

The above formal construction is the algebraization of the recursive computation of the Taylor coefficients of the formal inverse map $H^{-1}(x)$. Note that stabilization of truncations of the series (3.3) means that computation of the terms of any finite degree k of the components h_i' of the inverse map is achieved in a finite (depending on k) number of steps.

3C. Integration and formal flow of formal vector fields. Consider an (autonomous) formal ordinary differential equation

$$\dot{x}=F(x),\qquad F=(F_1,\dots,F_n)\in\mathcal{D}[[\mathbb{C}^n,0]]\cong\mathbb{C}[[x]]^n \tag{3.4}$$

with a *formal* right hand side part F. Since evaluation of a formal series at any point other than the origin makes no sense, the "standard" definition of solutions can at best be applied to constructing a solution with the initial condition $x(0)=0$. Yet in the most interesting case where $F(0)=0$, this solution is trivial, $x(t)\equiv 0$.

The alternative, suggested by Remark 1.20, is to define a *one-parametric subgroup of formal self-maps* $\{H^t\colon t\in\mathbb{C}\}\subset\operatorname{Diff}[[\mathbb{C}^n,0]]$ satisfying the condition

$$H^t\circ H^s=H^{t+s}\qquad\forall t,s\in\mathbb{C},\qquad H^0=E. \tag{3.5}$$

Together with the group $\{H^t\}$ of self-maps we always consider the corresponding one-parameter group of automorphisms $\{\mathbf{H}^t\}\subset\operatorname{Aut}\mathbb{C}[[x]]$.

This subgroup is said to be *holomorphic*, if all finite truncations j^kH^t depend holomorphically on t. For a holomorphic subgroup the derivative

$$\mathbf{F}=\left.\frac{d\mathbf{H}^t}{dt}\right|_{t=0}=\lim_{t\to0}t^{-1}(\mathbf{H}^t-\mathbf{E})\colon\mathbb{C}[[x]]\to\mathbb{C}[[x]] \tag{3.6}$$

is a formal vector field,

$$\begin{aligned}\mathbf{F}(fg) &= \frac{d}{dt}\bigg|_{t=0} \mathbf{H}^t(fg) = \frac{d}{dt}\bigg|_{t=0} \big[(\mathbf{H}^t f)(\mathbf{H}^t g)\big] \\ &= \left[\frac{d}{dt}\bigg|_{t=0} (\mathbf{H}^t f)\right](\mathbf{H}^0 g) + (\mathbf{H}^0 f)\left[\frac{d}{dt}\bigg|_{t=0} (\mathbf{H}^t g)\right] \\ &= g\,\mathbf{F}f + f\,\mathbf{F}g.\end{aligned}$$

Definition 3.8. A holomorphic one-parametric subgroup of formal self-maps $\{H^t\} \subseteq \operatorname{Diff}[[\mathbb{C}^n, 0]]$ is a *formal flow* of the formal vector field F corresponding to the derivation $\mathbf{F} \in \operatorname{Der}\mathbb{C}[[x]]$, if the corresponding group of automorphisms $\{\mathbf{H}^t\}$ satisfies the identity

$$\mathbf{F} = \frac{d\mathbf{H}^t}{dt}\bigg|_{t=0} \in \operatorname{Der}\mathbb{C}[[x]]. \tag{3.7}$$

The formal field F is called the *generator* of the subgroup $\{H^t\}$.

The above observation means that *any analytic one-parametric subgroup* of formal maps is always a formal flow of some formal field F (3.7). The following theorem is a formal analog of Proposition 1.19 showing that, conversely, *any* formal vector field F generates an holomorphic one-parametric subgroup of formal self-maps $\{H^t\} \subset \operatorname{Diff}[[\mathbb{C}, 0]]$.

Denote by $\mathbf{F}^m$ the iterated composition $\mathbf{F} \circ \cdots \circ \mathbf{F} \colon \mathbb{C}[[x]] \to \mathbb{C}[[x]]$ (m times) and consider the exponential series

$$\mathbf{H}^t = \exp t\mathbf{F} = \mathbf{E} + t\mathbf{F} + \frac{t^2}{2!}\,\mathbf{F}^2 + \cdots + \frac{t^m}{m!}\,\mathbf{F}^m + \cdots. \tag{3.8}$$

Theorem 3.9. *Any singular formal vector field F admits a formal flow $\{H^t\}$. This flow is defined by the series* (3.8) *which converges for all values of $t \in \mathbb{C}$ and depends analytically on t.*

Proof. We have to show that this series converges and its sum is an isomorphism of the algebra $\mathbb{C}[[x]]$ for any $t \in \mathbb{C}$. Then the identity (3.7) will follow automatically by the termwise differentiation of the series (3.8).

Convergence of the series (3.8) can be seen from the following argument. Let k be any finite order. Truncating the series (3.8), i.e., substituting $j^k\mathbf{F}$ instead of $\mathbf{F}$, we obtain a matrix formal power series. This series is always convergent: for an arbitrary choice of the norm $|\cdot|$ on the finite-dimensional space $J^k(\mathbb{C}^n, 0)$ the norm of the operator $j^k\mathbf{F}$ is finite, $|j^k\mathbf{F}| = r < +\infty$, and hence the series (3.8) is majorized by the convergent scalar series $1 + |t|r + |t|^2r^2/2! + \cdots = \exp|t|r < +\infty$ for any finite $t \in \mathbb{C}$; cf. with Definition 1.7. Denote its sum by $\exp j^k\mathbf{F} \colon J^k(\mathbb{C}^n, 0) \to J^k(\mathbb{C}^n, 0)$.

Truncations $\exp j^k\mathbf{F}$ for different orders k agree in common terms: if $l > k$, then $j^k(\exp t\, j^l\mathbf{F}) = \exp t\, j^k\mathbf{F}$. This allows us to define the sum of the series $\exp t\mathbf{F}$ as a linear operator $\mathbf{H}^t\colon \mathbb{C}[[x]] \to \mathbb{C}[[x]]$ via its finite truncations of all orders.

The group property $\mathbf{H}^{t+s} = \mathbf{H}^t \circ \mathbf{H}^s$ equivalent to the group property (3.5), follows from the formal identity $\exp(t+s) = \exp t \cdot \exp s$, since $t\mathbf{F}$ and $s\mathbf{F}$ obviously commute. It remains to show that $\mathbf{H}^t$ is an algebra *homomorphism*, i.e., $\mathbf{H}^t(fg) = \mathbf{H}^t f\, \mathbf{H}^t g$ for any two series $f, g \in \mathbb{C}[[x]]$.

By the iterated Leibnitz rule, for any $f, g \in \mathbb{C}[[x]]$,

$$\mathbf{F}^k(fg) = \sum_{p+q=k} \tfrac{(p+q)!}{p!q!}\, \mathbf{F}^p f \cdot \mathbf{F}^q g.$$

Substituting this identity into the exponential series, we have

$$\begin{aligned}\mathbf{H}^t(fg) &= \sum_k \tfrac{t^k}{k!}\, \mathbf{F}^k(fg) = \sum_k \sum_{p+q=k} \tfrac{t^{p+q}}{p!q!}\, \mathbf{F}^p f \cdot \mathbf{F}^q g \\ &= \Big(\sum_p \tfrac{t^p}{p!}\, \mathbf{F}^p f\Big) \cdot \Big(\sum_q \tfrac{t^q}{q!}\, \mathbf{F}^q g\Big) = \mathbf{H}^t f \cdot \mathbf{H}^t g. \quad \square\end{aligned}$$

Motivated by the series (3.8), we will often use the exponential notation: if F is a formal or analytic vector field with a singular point at the origin, we will denote by $\exp tF$ the time t flow (formal or analytic) of this field.

3D. Embedding in the flow and matrix logarithms.

Definition 3.10. A holomorphic germ $H \in \operatorname{Diff}(\mathbb{C}^n, 0)$ or a formal self-map $H \in \operatorname{Diff}[[\mathbb{C}^n, 0]]$ is said to be *embeddable*, if there exists a holomorphic germ of a vector field F (resp., a formal vector field $F \in \mathcal{D}[[\mathbb{C}^n, 0]]$) such that H is a time one (resp., formal time one) flow map of F, i.e., $H = \exp F$.

For a linear system $\dot x = Ax$ with constant coefficients, the flow consists of *linear* maps $x \mapsto (\exp tA)x$; see (1.12). Therefore for a *linear* map $x \mapsto Mx$, $M \in \operatorname{GL}(n, \mathbb{C})$, it is natural to consider the embedding problem in the class of linear vector fields $F(x) = Ax$. Then the problem reduces to finding a *matrix logarithm*, a matrix solution of the equation

$$\exp A = M, \qquad A, M \in \operatorname{Mat}(n, \mathbb{C}). \tag{3.9}$$

Clearly, the necessary condition for solvability of this equation is *nondegeneracy* of M. It also turns out to be sufficient for matrices over the field $\mathbb{C}$.

Lemma 3.11. *For any nondegenerate matrix $M \in \operatorname{Mat}(n, \mathbb{C})$, $\det M \neq 0$, there exists the matrix logarithm $A = \ln M$, a complex matrix satisfying the equation* (3.9)

Proof. We give two constructions of matrix logarithms for nondegenerate matrices.

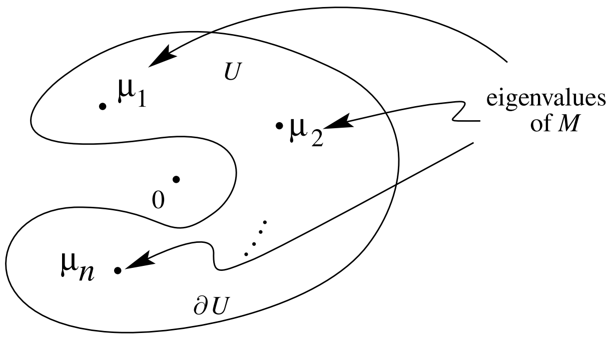

Figure I.3. Construction of the integral representation of the matrix logarithm for a nondegenerate matrix with the given spectrum

1. First, for a scalar matrix $M = \lambda E$, $0 \neq \lambda \in \mathbb{C}$, the logarithm can be defined as $\ln M = (\ln \lambda)\, E$, for any choice of $\ln \lambda$. A matrix having a single nonzero eigenvalue of high multiplicity has the form $M = \lambda(E + N)$, where N is a nilpotent (upper-triangular) matrix. Its logarithm can be defined using the formal series for the scalar logarithm as follows:

$$\ln M = \ln(\lambda E) + \ln(E + N) = (\ln \lambda)\, E + N - \tfrac{1}{2}N^2 + \tfrac{1}{3}N^3 - \cdots \qquad (3.10)$$

(the sum is finite). This formula gives a well-defined answer by virtue of the formal identity $\exp(x - \frac{x^2}{2} + \frac{x^3}{3} \pm \dots) = 1 + x$, since the matrices E and N commute.

An arbitrary matrix M can be reduced to a block diagonal form with each block having a single eigenvalue. The block diagonal matrix formed by logarithms of individual blocks solves the problem of computing the matrix logarithm in the general case.

2. The second proof uses the integral representation for analytic matrix functions. For any function $f(x)$ complex analytic in a domain $U \subset \mathbb{C}$ bounded by a simple curve ∂U and any matrix M with all eigenvalues in U, the value $f(M)$ can be defined by the contour integral

$$f(M) = \frac{1}{2\pi i} \oint_{\partial U} f(\lambda)(\lambda E - M)^{-1}\, d\lambda \qquad (3.11)$$

[**Gan59**, Ch. V, §4]. In application to $f(x) = \ln x$ we have to choose a simple loop ∂U containing all eigenvalues of M inside U but the origin $\lambda = 0$ outside (cf. with Fig. I.3). Then in the domain U one can unambiguously select a branch of complex logarithm $\ln \lambda$ which can be substituted into the integral representation.

To prove that the integral representation gives the same answer as before, it is sufficient to verify it only for the diagonal matrices, when the inverse can be computed explicitly. The advantage of this formula is the possibility of bounding the norm $|\ln M|$ defined by the above integral, in terms of $|M|$ and $|M^{-1}|$. □

Remark 3.12. The matrix logarithm is *by no means unique*. In the first construction one has the freedom to choose branches of logarithms of eigenvalues arbitrarily and independently for different Jordan blocks. In the second construction besides choosing the branch of the logarithm, there exists a freedom to choose the domain U (i.e., the loop ∂U encircling all the eigenvalues of M but not the origin).

Remark 3.13. There is a natural obstruction for extracting the matrix logarithm in the class of *real* matrices. If $\exp A = M$ for some real matrix A, then M can be connected with the identity E by a path of nondegenerate matrices $\exp tA$, in particular, M should be orientation-preserving. If M is nondegenerate but orientation-reverting, it has no real matrix logarithm.

However, there are more subtle obstructions. Consider the real matrix $M = \begin{pmatrix} -1 & 1 \\ & -1 \end{pmatrix}$ with determinant 1. If $M = \exp A$, then by (1.16) $\exp \operatorname{tr} A = 1$ so that for a real matrix necessarily $\operatorname{tr} A = 0$. The two eigenvalues cannot be simultaneously zero, since then the exponent will have the eigenvalues both equal to 1. Therefore the eigenvalues must be different, in which case the matrix A and hence its exponent M must be diagonalizable. The contradiction shows impossibility of solving the equation $\exp A = M$ in the class of real matrices.

3E. Logarithms and derivations. Inspired by the construction of the matrix exponential, one can attempt to prove that for any formal map $H \in \operatorname{Diff}[[\mathbb{C}^n, 0]]$ there exists a formal vector field F whose formal time one flow coincides with H, as follows.

Consider an arbitrary finite order k and the k-jet $\mathbf{H}_k = j^k\mathbf{H}$ considered as an isomorphism of the finite-dimensional $\mathbb{C}$-algebra $\mathfrak{F}^k = J^k(\mathbb{C}^n, 0)$. By Lemma 3.11, there exists a linear map $\mathbf{F}_k \colon \mathfrak{F}^k \to \mathfrak{F}^k$ such that $\exp \mathbf{F}_k = \mathbf{H}_k$.

Assume that for some reasons

(i) jets of the logarithms $\mathbf{F}_k$ of different orders agree after truncation, i.e., $j^k\mathbf{F}_l = \mathbf{F}_k$ for $l > k$, and

(ii) each $\mathbf{F}_k$ is a *derivation* of the commutative algebra $\mathfrak{F}^k$, thus a k-jet of a vector field.

Then together these jets would define a derivation $\mathbf{F}$ of the algebra $\mathfrak{F} = \mathbb{C}[[x]]$.

The first objective can be achieved if $\mathbf{F}_k$ are truncations of some polynomial or infinite series. There is only one such candidate, the *logarithmic series* $\ln \mathbf{H} \colon \mathbb{C}[[x]] \to \mathbb{C}[[x]]$, obtained from the formal series for

$\ln(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 \mp \cdots$ by substitution,

$$\ln \mathbf{H} = (\mathbf{H} - \mathbf{E}) - \frac{1}{2}(\mathbf{H} - \mathbf{E})^2 + \frac{1}{3}(\mathbf{H} - \mathbf{E})^3 \mp \cdots \qquad (3.12)$$

(cf. with (3.10)). Until the end of this section we use the notation $\ln \mathbf{H}$ only in the sense of the series (3.12).

The series for $\ln \mathbf{H}$ does not converge everywhere even in the finite-dimensional case: the domain of convergence contains the ball $|\mathbf{H} - \mathbf{E}| < 1$ and all unipotent finite-dimensional matrices, but most certainly *not* the matrix $-\mathbf{E}$. Besides that difficulty, it is absolutely not clear why the formal logarithm of an isomorphism of the commutative algebra $\mathbb{C}[[x]]$, even if it converges, must be a derivation: no simple arguments similar to the one used in the proof of Theorem 3.9, exist (sometimes this circumstance is overlooked).

Let $\mathfrak{F}$ be a commutative $\mathbb{C}$-algebra of finite dimension n over $\mathbb{C}$ and $\mathbf{H}$ an automorphism of $\mathfrak{F}$.

Theorem 3.14. *The series* (3.12) *converges for all unipotent automorphisms* $\mathbf{H}$ *of a finite dimensional algebra* $\mathfrak{F}$ *and its sum* $\mathbf{F} = \ln \mathbf{H}$ *in this case is a derivation of this algebra.*

Proof using the Lie group tools. Consider the matrix Lie group $\mathfrak{T} \subset \mathrm{GL}(n, \mathbb{C})$ of upper-triangular matrices with units on the principal diagonal and the corresponding Lie algebra $\mathfrak{t} \subset \mathrm{Mat}(n, \mathbb{C})$ of *strictly* upper-triangular matrices.

The exponential series (3.8) and the matrix logarithm (3.12) restricted on $\mathfrak{t}$ and $\mathfrak{T}$ respectively, are *polynomial* maps defined everywhere. They are mutually inverse: for any $\mathbf{F} \in \mathfrak{t}$ and $\mathbf{H} \in \mathfrak{T}$ we have $\ln \exp \mathbf{F} = \mathbf{F}$ and $\exp \ln \mathbf{H} = \mathbf{H}$. This follows from the identities $\ln e^z = z$, $e^{\ln w} = w$ expanded in the series. In particular, exp is surjective.

For any Lie subalgebra $\mathfrak{g} \subseteq \mathfrak{t}$ and the corresponding Lie subgroup $\mathfrak{G} \subseteq \mathfrak{T}$ the exponential map $\exp\colon \mathfrak{g} \to \mathfrak{G}$ is the restriction of (3.8) on $\mathfrak{g}$.

By [**Var84**, Theorem 3.6.2], the exponential map remains surjective also on $\mathfrak{G}$, i.e., $\exp \mathfrak{g} = \mathfrak{G}$. We claim that in this case the logarithm maps $\mathfrak{G}$ *into* $\mathfrak{g}$. Indeed, if $\mathbf{H} \in \mathfrak{G}$ and $\mathbf{H} = \exp \mathbf{F}$ for some $\mathbf{F} \in \mathfrak{g}$, then $\ln \mathbf{H} = \ln \exp \mathbf{F} = \mathbf{F} \in \mathfrak{g}$.

The assertion of the theorem arises if we take $\mathfrak{G} = \mathfrak{T} \cap \mathrm{Aut}(\mathfrak{F})$ to be the Lie subgroup of *triangular automorphisms* of $\mathfrak{F} \cong \mathbb{C}^n$ and $\mathfrak{g} = \mathfrak{t} \cap \mathrm{Der}(\mathfrak{F})$ of *triangular derivations* of the commutative algebra $\mathfrak{F}$. □

Remark 3.15. Surjectivity of the exponential map for a subgroup of the triangular group $\mathfrak{T}$ is a delicate fact that follows from the nilpotency of the Lie algebra $\mathfrak{t}$. Indeed, by the Campbell–Hausdorff formula, $\exp \mathbf{F} \cdot \exp \mathbf{G} =$

$\exp \mathbf{K}$, where $\mathbf{K} = \mathbf{K}(\mathbf{F}, \mathbf{G})$ is a series which in the nilpotent case is a polynomial map $\mathfrak{t} \times \mathfrak{t} \to \mathfrak{t}$ defined everywhere. Thus the image $\exp \mathfrak{g}$ is a *Lie subgroup* in $\mathfrak{G} \subseteq \mathfrak{T}$ for *any* subalgebra $\mathfrak{g}$, containing a small neighborhood of the unit $\mathbf{E}$. It is well known that any such neighborhood generates (by the group operation) the whole connected component of the unit, so that $\exp \mathfrak{g}$ coincides with this component. If $\mathfrak{G}$ is simply connected, then $\exp \mathfrak{g} = \mathfrak{G}$ as asserted.

Without nilpotency the answer may be different: as follows from Remark 3.13, for two Lie algebras $\mathfrak{gl}(n, \mathbb{R}) \subset \mathfrak{gl}(n, \mathbb{C})$ and the respective Lie groups $\mathrm{GL}(n, \mathbb{R}) \subset \mathrm{GL}(n, \mathbb{C})$, the exponent is surjective on the ambient (bigger) group but *not* on the subgroup.

Remark 3.16. Using similar arguments, one can prove that for an arbitrary automorphism $\mathbf{H} \in \mathrm{Aut}(\mathfrak{F})$ *sufficiently close to the unit* $\mathbf{E}$, the logarithm $\ln \mathbf{H}$ given by the series (3.12) is a derivation, $\ln \mathbf{H} \in \mathrm{Der}(\mathfrak{F})$. This follows from the fact that ln and exp are mutually inverse on sufficiently small neighborhoods of $\mathbf{E}$ and 0 respectively. However, the size of this neighborhood depends on $\mathfrak{F}$.

3F. Embedding in the formal flow. Based on Theorem 3.14, one can prove the following general result obtained by F. Takens in 1974; see [**Tak01**].

Theorem 3.17. *Let $H \in \mathrm{Diff}[[\mathbb{C}^n, 0]]$ be a formal map whose linearization matrix $A = \frac{\partial H}{\partial x}(0)$ is unipotent, $(A - E)^n = 0$.*

Then there exists a formal vector field $F \in \mathcal{D}[[\mathbb{C}^n, 0]]$ whose linearization is a nilpotent matrix N, such that H is the formal time 1 map of F.

Proof. As usual, we identify the formal map with an automorphism $\mathbf{H}$ of the algebra $\mathfrak{F} = \mathbb{C}[[x_1, \dots, x_n]]$ so that its finite k-jets $j^k\mathbf{H}$ become automorphisms of the finite dimensional algebras $\mathfrak{F}^k = J^k(\mathbb{C}^n, 0)$. Without loss of generality we may assume that the matrix A is upper-triangular so that the isomorphism $\mathbf{H}$ and all its truncations $j^k\mathbf{H}$ in the canonical `deglex`-ordered basis becomes upper-triangular with units on the diagonal: the jets $j^k\mathbf{H}$ are finite-dimensional upper-triangular (unipotent) automorphisms of the algebras $\mathfrak{F}^k$.

Consider the infinite series (3.12) together with its finite-dimensional truncations obtained by applying the operation j^k to all terms. Each such truncation is a logarithmic series for $\ln j^k\mathbf{H}$ which converges (actually, stabilizes after finitely many steps) and its sum is a derivation $j^k\mathbf{F}$ of $\mathfrak{F}^k$ by Theorem 3.14. Clearly, different truncations agree on the lower order terms, thus $\ln \mathbf{H}$ converges in the sense of Definition 3.4 to a derivation $\mathbf{F}$ of $\mathfrak{F}$. This derivation corresponds to the formal vector field F as required. □

Exercises and Problems for §3.

Problem 3.1. Let $F \in \mathcal{D}[[\mathbb{C}^n, 0]]$ be a formal vector field corresponding to the derivation $\mathbf{F} \in \operatorname{Der}\mathbb{C}[[x]]$, and $\{H^t\} \subset \operatorname{Diff}[[\mathbb{C}^n, 0]]$ its formal flow corresponding to the one-parametric group of automorphisms $\{\mathbf{H}^t\} \subset \operatorname{Aut}\mathbb{C}[[x]]$ related by the identity (3.7).

Prove that in this case $\frac{d}{dt}H^t = F \circ H^t$ for any t on the level of vector formal series.

Exercise 3.2. Consider the derivation $\mathbf{F} = \frac{\partial}{\partial x}$ on the algebra $\mathbb{C}[x]$ of univariate polynomials. Prove that the exponential series $\exp t\mathbf{F}$ is well defined for all values of $t \in \mathbb{C}$ as an automorphism of $\mathbb{C}[x]$, but is not defined if the algebra $\mathbb{C}[x]$ is replaced by the algebras $\mathbb{C}[[x]]$ or $\mathcal{O}(\mathbb{D})$, where $\mathbb{D} = \{|x| < 1\}$ is the unit disk.

Problem 3.3. Prove that the integral representation (3.11) coincides with the standard definition of a matrix function $f(M)$ in the case where f is a (scalar) polynomial.

Exercise 3.4. Find *all* complex logarithms of the matrix $M = \begin{pmatrix} -1 & 1 \\ & -1 \end{pmatrix}$ (i.e., solutions of the equation $\exp A = M$).

4. Formal normal forms

In the same way as holomorphic maps act on holomorphic vector fields by conjugacy (1.26), formal maps act on formal vector fields. In this section we investigate the *formal normal forms*, to which a formal vector field can be brought by a suitable formal isomorphism.

Definition 4.1. Two formal vector fields F, F' are *formally equivalent*, if there exists an invertible formal self-map H such that the identity (1.26) holds on the level of formal series.

The fields are formally equivalent if and only if the corresponding derivations $\mathbf{F}, \mathbf{F}'$ of the algebra $\mathbb{C}[[x]]$ are conjugated by a suitable isomorphism $\mathbf{H} \in \operatorname{Diff}[[\mathbb{C}^n, 0]]$ of the formal algebra: $\mathbf{H} \circ \mathbf{F}' = \mathbf{F} \circ \mathbf{H}$.

Obviously, two holomorphically equivalent (holomorphic) germs of vector fields are formally equivalent. The converse is in general not true, as the formal self-maps may be divergent.

4A. Formal classification theorem. Formal classification of formal vector fields strongly depends on its principal part, in particular, on properties of the linearization matrix $A = \left(\frac{\partial F}{\partial x}\right)(0)$ when the latter is nonzero (cases with $A = 0$ are hopelessly complicated if the dimension is greater than one).

We start with the most important example and introduce the definition of a resonance as a certain arithmetic (i.e., involving integer coefficients) relation between complex numbers.