5. Holomorphic normal forms

5A. Poincaré and Siegel domains. To linearize a given (say, nonresonant) vector field, on each step of the Poincaré–Dulac process one has to compute the inverse of the operator $\operatorname{ad}_A = [\mathbf{A}, \cdot\,]$ on the spaces of homogeneous vector fields. To that end, one has to divide the Taylor coefficients by the *denominators*, expressions of the form $\lambda_j - \langle \alpha, \lambda\rangle \in \mathbb{C}$ with $\alpha \in \mathbb{Z}_+^n$, $|\alpha| \geqslant 2$, that may a priori be *small* even in the nonresonant case where ad_A is invertible. These denominators associated with the spectrum λ of the linearization matrix A, behave differently as $|\alpha|$ grows to infinity, in the following two different cases.

Definition 5.1. The *Poincaré domain* $\mathfrak{P} \subset \mathbb{C}^n$ is the collection of all tuples $\lambda = (\lambda_1, \dots, \lambda_n)$ such that the convex hull of the point set $\{\lambda_1, \dots, \lambda_n\} \subset \mathbb{C}$ does not contain the origin inside or on the boundary.

The Siegel domain $\mathfrak{S}$ is the complement to the Poincaré domain in $\mathbb{C}^n$.

The *strict Siegel domain* is the set of tuples for which the convex hull contains the origin strictly inside.

Sometimes we say about *tuples*, *spectra* or even germs of vector fields at singular points as being of Poincaré (resp., Siegel) type.

Proposition 5.2. *If λ is of Poincaré type, then only finitely many denominators $\lambda_j - \langle \alpha, \lambda\rangle$, $\alpha \in \mathbb{Z}_+^n$, $|\alpha| \geqslant 2$, may actually vanish.*

Moreover, nonzero *denominators are bounded away from the origin: the latter is an isolated point of the set of all denominators $\{\lambda_j - \langle \alpha, \lambda\rangle \mid j = 1, \dots, n,\ |\alpha| \geqslant 2\}$.*

On the contrary, if λ is of Siegel type, then either there are infinitely many vanishing denominators, or the origin $0 \in \mathbb{C}$ is their accumulation point (these two possibilities are not mutually exclusive).

Proof. If the convex hull of $\{\lambda_1, \dots, \lambda_n\} \subset \mathbb{C}$ does not contain the origin, by the convex separation theorem there exists a real linear functional $\ell \colon \mathbb{C}^2 \to \mathbb{R}$ such that $\ell(\lambda_j) \leqslant -r < 0$ for all λ_j, and hence $\ell(\langle \alpha, \lambda\rangle) \leqslant -r|\alpha|$. But then for any denominator we have

$$\ell(\lambda_j - \langle \alpha, \lambda\rangle) \geqslant \ell(\lambda_j) + |\alpha| r \to +\infty \qquad \text{as } |\alpha| \to \infty.$$

Since ℓ is bounded on any small neighborhood of the origin $0 \in \mathbb{C}$, the first two assertions are proved.

To prove the last assertion, notice that in the Siegel case there are either two or three numbers, whose linear combination with positive (real) coefficients is zero, depending on whether the origin lies on the boundary or in the interior of the convex hull. We give the proof in the second case, more difficult and more generic (the proof for the first case is simpler).

If the origin lies inside a triangle formed by the eigenvalues, then modulo their re-enumeration and a (nonconformal) affine transformation of the complex plane $\mathbb{R}^2 \cong \mathbb{C}$, we may assume without loss of generality that $\lambda_1 = 1$, $\lambda_2 = +i$ and $-\lambda_3 \in \mathbb{R}^2_+ = \mathbb{R}_+ + i\mathbb{R}_+$. In this case all "fractional parts" $-\mathbb{N}\lambda_3 \bmod \mathbb{Z} + i\mathbb{Z}$ of natural multiples of $-\lambda_3$ either form a finite subset of the 2-torus $\mathbb{R}^2/\mathbb{Z}^2$ (in which case all points of this set correspond to infinitely many vanishing denominators), or are uniformly distributed along some 1-torus, or dense. In both latter cases the point $(0,0) \in \mathbb{R}^2/\mathbb{Z}^2$ is the accumulation point of the "fractional parts" which are affine images of the denominators. □

Corollary 5.3. *If the spectrum of the linearization matrix A of a formal vector field belongs to the Poincaré domain, then the resonant formal normal form for this field established in Theorem 4.10, is polynomial.* □

Remark 5.4. Resonant tuples $\lambda \in \mathbb{C}^n$ are dense in the Siegel domain $\mathfrak{S}$ and not dense in the Poincaré domain $\mathfrak{P}$. The proof of this fact can be found in [**Arn83**].

5B. Holomorphic classification in the Poincaré domain. In the Poincaré domain, the normalizing series reducing vector fields or holomorphic maps to their Poincaré–Dulac normal forms, always converge.

Theorem 5.5 (Poincaré normalization theorem)**.** *A holomorphic vector field with the linear part of Poincaré type is holomorphically equivalent to its polynomial Poincaré–Dulac formal normal form.*

In particular, if the field is nonresonant, then it can be linearized by a holomorphic transformation.

We prove this theorem first for vector fields with a diagonal nonresonant linear part $\varLambda = \operatorname{diag}\{\lambda_1, \dots, \lambda_n\}$. The resonant case will be addressed later in §5**C**. The classical proof by Poincaré was achieved by the so-called *majorant method.* In the modern language, it takes a more convenient form of the contracting map principle in an appropriate functional space, the *majorant space.*

Definition 5.6. The *majorant operator* is the nonlinear operator acting on formal series by replacing all Taylor coefficients by their absolute values,

$$\mathbf{M}\colon \sum_{\alpha\in\mathbb{Z}^+_n} c_\alpha\, z^\alpha \mapsto \sum_{\alpha\in\mathbb{Z}^+_n} |c_\alpha|\, z^\alpha.$$

The action of the majorant operator naturally extends on all formal objects (vector formal series, formal vector fields, formal transformations, *etc.*).

Definition 5.7. The *majorant ρ-norm* is the functional on the space of formal power series $\mathbb{C}[[z_1,\dots,z_n]]$, defined as

$$[\![f]\!]_\rho = \sup_{|z|<\rho} |\mathbf{M} f(z)| = |\mathbf{M}f(\rho,\dots,\rho)| \leqslant +\infty. \tag{5.1}$$

For a formal vector function $F=(F_1,\dots,F_n)$ the majorant norm is

$$[\![F]\!]_\rho = [\![F_1]\!]_\rho + \dots + [\![F_n]\!]_\rho \,. \tag{5.2}$$

The majorant space $\mathcal{B}_\rho$ is the subspace of formal (vector) functions from $\mathbb{C}[[x]]$ having finite majorant ρ-norm.

Proposition 5.8. *The space $\mathcal{B}_\rho$ with the majorant norm $[\![\cdot]\!]_\rho$ is complete.*

Proof. If $\rho=1$, this is obvious: $\mathcal{B}_1$ is the space of infinite absolutely converging sequences $\{c_\alpha\}$, and hence is isomorphic to the standard Lebesgue space ℓ^1 which is complete. The general case of an arbitrary ρ follows from the fact that the correspondence $f(\rho x) \leftrightarrow f(x)$ is an isomorphism between $\mathcal{B}_\rho$ and $\mathcal{B}_1$. ☐

Remark 5.9. The space $\mathcal{B}_\rho$ is closely related but not coinciding with the space $\mathcal{A}_\rho = \mathcal{A}(D_\rho)$ of functions, holomorphic in the polydisk $D_\rho = \{|z|<\rho\}$, continuous on its closure and equipped with the usual sup-norm $\|f\|_\rho = \max_{|z|<\rho} |f(z)|$.

Clearly, $\mathcal{B}_\rho \subset \mathcal{A}_\rho$, since a series from $\mathcal{B}_\rho$ is absolutely convergent on the closed polydisk $\overline{D_\rho}$. Conversely, if f is holomorphic in D_ρ and continuous on the boundary, then by the Cauchy estimates, the Taylor coefficients c_α of f satisfy the inequality

$$|c_\alpha| \leqslant \|f\|_\rho \cdot \rho^{-|\alpha|}, \quad \alpha \in \mathbb{Z}_+^n.$$

Though the series $[\![f]\!]_\rho = \sum |c_\alpha|\,\rho^{|\alpha|}$ may still diverge, any other norm $[\![f]\!]_{\rho'}$ with $\rho'<\rho$, will already be finite:

$$[\![f]\!]_{\rho'} \leqslant \|f\|_\rho \cdot \sum_{\alpha\in\mathbb{Z}_+^n} \delta^{|\alpha|} < C\,\|f\|_\rho, \qquad C=C(\delta,n), \quad \delta=\rho'/\rho<1.$$

To construct a counterexample showing that indeed $\mathcal{A}_\rho \supsetneqq \mathcal{B}_\rho$, consider a convergent but not absolutely convergent Fourier series $\sum_{k\in\mathbb{Z}_+} c_k e^{ikt}$ in one real variable t and let $f(z)=\sum c_k z^k$. Such a series converges at all points of the boundary $|z|=1$ and represents a function from $\mathcal{A}(D_1)$, but by construction its 1-norm is infinite. Details can be found in [**Edw79**, §10.6]

The important properties of the majorant spaces and norms concern operations on functions. We will use the notation $f \ll g$ for two vector series from $\mathbb{C}^n[[x]]$ with positive coefficients, if each coefficient of f is no greater than the corresponding coefficient of g. In a similar way the notation $x \ll y$ will be used to denote the componentwise set of inequalities between two vectors $x, y \in \mathbb{R}^n$. If $f \in \mathbb{R}^n[[x]]$ is a (vector) series with *nonnegative* coefficients, then it is monotonous: $f(x) \ll f(y)$ if $x \ll y$.

Lemma 5.10. *1. For any two series $f, g \in \mathbb{C}[[x]]$ and any ρ,*

$$\llbracket fg \rrbracket_\rho \leqslant \llbracket f \rrbracket_\rho \cdot \llbracket g \rrbracket_\rho, \tag{5.3}$$

provided that all norms are finite.

2. If $G \ll G'$, are two formal series from $\mathbb{R}^n[[x]]$ and F is a series with nonnegative coefficients, then $F \circ G \ll F \circ G'$.

3. If $F, G \in \mathbb{C}^n[[z_1, \dots, z_n]]$ are two formal vector series, $F(0) = G(0) = 0$, then for their composition we have

$$\llbracket F \circ G \rrbracket_\rho \leqslant \llbracket F \rrbracket_\sigma, \qquad \sigma = \llbracket G \rrbracket_\rho. \tag{5.4}$$

Proof. The first two statements are obvious: all Taylor coefficients of the product or composition are obtained from the coefficients of entering terms by operations of addition and multiplication only. In particular, $\mathbf{M}(fg) \ll \mathbf{M}\, f \cdot \mathbf{M}\, g$. Evaluating both parts at $\boldsymbol{\rho} = (\rho, \dots, \rho)$ proves the first statement.

Since all binomial coefficients are nonnegative (in fact, natural numbers), we have $\mathbf{M}(F \circ G) \ll (\mathbf{M}\, F) \circ (\mathbf{M}\, G)$. Evaluating at $\boldsymbol{\rho} = (\rho, \dots, \rho)$ yields $\mathbf{M}\, G(\boldsymbol{\rho}) = y \ll \boldsymbol{\sigma} = (\sigma, \dots, \sigma)$, where $\sigma = \llbracket G \rrbracket_\rho$. By monotonicity, $\|F \circ G\|_\rho = \big((\mathbf{M}\, F) \circ (\mathbf{M}\, G)\big)(\boldsymbol{\rho}) \ll \mathbf{M}\, F(y) \ll \mathbf{M}\, F(\boldsymbol{\sigma}) = \llbracket F \rrbracket_\sigma$. The last statement is proved. ☐

Lemma 5.11. *If $\varLambda \in \operatorname{Mat}(n, \mathbb{C})$ is a nonresonant diagonal matrix of Poincaré type, then the operator $\operatorname{ad}_\varLambda$ has a bounded inverse in the space of vector fields equipped with the majorant norm.*

Proof. The formal inverse operator $\operatorname{ad}_\varLambda^{-1}$ is diagonal,

$$\operatorname{ad}_\varLambda^{-1} \colon \sum_{k,\alpha} c_{k\alpha} x^\alpha \frac{\partial}{\partial x_k} \longmapsto \sum_{k,\alpha} \frac{c_{k\alpha}}{\lambda_k - \langle \alpha, \lambda \rangle}\, x^\alpha \frac{\partial}{\partial x_k}.$$

In the Poincaré domain the absolute values of all denominators are bounded from below by a positive constant $\varepsilon > 0$, therefore *any* majorant ρ-norm is increased by no more than ε^{-1}:

$$\llbracket \operatorname{ad}_\varLambda^{-1} \rrbracket_\rho \leqslant \left(\inf_{j,\alpha} |\lambda_j - \langle \alpha, \lambda \rangle\,| \right)^{-1} < +\infty.$$

This proves that $\operatorname{ad}_\varLambda$ has the bounded inverse. ☐

Remark 5.12. A diagonal operator of the form $\sum_\alpha c_\alpha z^\alpha \mapsto \sum \mu_\alpha c_\alpha x^\alpha$ with bounded entries, $\sup_\alpha |\mu_\alpha| < +\infty$, which is always defined and bounded in the majorant norm, may be *not defined* or defined but unbounded on the holomorphic space $\mathcal{A}(D_\rho)$; see Remark 5.9. The "real" counterexample is even simpler: the operator which multiplies odd coefficients by -1, sends the series $1 - x^2 + x^4 - \cdots$, converging and bounded on $[-1, 1]$, into an unbounded function.

Let $F = (F_1, \dots, F_n) \in \mathcal{D}(\mathbb{C}^n, 0)$ be a holomorphic vector function defined in some polydisk near the origin. The *operator of argument shift* is the operator

$$S_F \colon h(x) \mapsto F(x + h(x)), \tag{5.5}$$

acting on holomorphic vector fields $h \in \mathcal{D}(\mathbb{C}^n, 0)$ without the free term, $h(0) = 0$. We want to show that S_F is in some sense strongly contracting. The formal statement looks as follows.

Consider the one-parameter family of majorant Banach spaces $\mathcal{B}_\rho$ as in Definition 5.7 indexed by the real parameter $\rho \in (\mathbb{R}_+, 0)$. We consider $\mathcal{B}_{\rho'}$ as a subspace in $\mathcal{B}_\rho$ for all $0 < \rho < \rho'$ (the natural embedding $\mathrm{id}_{\rho',\rho} \colon \mathcal{B}_{\rho'} \to \mathcal{B}_\rho$ is continuous).

Let S be an operator defined on all of these spaces for all sufficiently small values of ρ, considered as a family of operators $S_\rho \colon \mathcal{B}_\rho \to \mathcal{B}_\rho$ which commute with the "restriction operators" $\mathrm{id}_{\rho',\rho}$ for any $\rho < \rho'$, but we will omit the subscript in the notation of $S_\rho = S$.

Definition 5.13. The operator $S \cong \{S_\rho\}$ is *strongly contracting*, if

1. $\llbracket S(0) \rrbracket_\rho = O(\rho^2)$ and
2. S is Lipschitz on the ball $B_\rho = \{\llbracket h \rrbracket_\rho \leqslant \rho\} \subset \mathcal{B}_\rho$ of the majorant ρ-norm (with the same ρ), with the Lipschitz constant no greater than $O(\rho)$ as $\rho \to 0$.

Note that any strongly contracting operator takes the balls B_ρ strictly into themselves, since the center of the ball is shifted by $O(\rho^2)$ and the diameter of the image $S(B_\rho)$ does not exceed $2\rho\, O(\rho) = O(\rho^2)$.

The involved definition of strong contraction intends to make the formulation of the following claim easy.

Lemma 5.14. *Assume that the germ $F \colon (\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$ is holomorphic and its linearization is zero, $\left(\frac{\partial F}{\partial x}\right)(0) = 0$.*

Then the operator of argument shift (5.5) *is strongly contracting.*

Proof. First note that S_F takes $h = 0$ into $F(x)$; the latter function has ρ-norm $O(\rho^2)$ for all sufficiently small ρ, since F begins with quadratic terms.

Next we compute the Lipschitz constant for $S = S_F$ restricted on the ball $B_\rho \subseteq \mathcal{B}_\rho$. If $h, h' \in \mathbb{C}^n[[x_1, \dots, x_n]]$ are two vector fields, then the difference

$$g = Sh - Sh' = F \circ (\mathrm{id} + h) - F \circ (\mathrm{id} + h')$$

can be represented as the integral

$$g(x) = \int_0^1 \left(\frac{\partial F}{\partial x}\right) \bigl(x + \tau h(x) + (1-\tau)h'(x)\bigr) \cdot (h(x) - h'(x))\, d\tau.$$

By Lemma 5.10, since $\tau \in [0,1]$, we have

$$\llbracket g \rrbracket_\rho \leqslant \llbracket \tfrac{\partial F}{\partial x} \rrbracket_\sigma \cdot \llbracket h - h' \rrbracket_\rho, \qquad \sigma = \llbracket x + \tau h(x) + (1-\tau)h'(x) \rrbracket_\rho.$$

The norm σ is no greater than $\llbracket x \rrbracket_\rho + \max(\llbracket h \rrbracket_\rho, \llbracket h' \rrbracket_\rho) = (n+1)\rho$ if both h, h' are from the ρ-ball B_ρ. On the other hand, if F is a holomorphic vector function without free and linear terms, its Jacobian matrix is holomorphic without free terms and hence its σ-norm is no greater than $C\sigma$ for all sufficiently small $\sigma > 0$. Collecting everything together, we see that S_F is Lipschitz on the ρ-ball B_ρ, with the Lipschitz constant (contraction rate) not exceeding $(n+1)C\rho$, so S_F is strongly contracting. ☐

Proof of Theorem 5.5 (nonresonant case). Now we can prove that a holomorphic vector field with diagonal nonresonant linearization matrix Λ of Poincaré type is holomorphically linearizable in a sufficiently small neighborhood of the origin. The proof serves as a paradigm for a more technically involved proof required for the resonant case.

A holomorphic transformation $H = \operatorname{id} + h$ conjugates the linear vector field Λx (the normal form) with the initial nonlinear field $\Lambda x + F(x)$, if and only if

$$\Lambda h(x) - \bigl(\tfrac{\partial h}{\partial x}\bigr)\, \Lambda x = F\bigl(x + h(x)\bigr). \tag{5.6}$$

Using the operators introduced earlier, this can be rewritten as the identity

$$\operatorname{ad}_\Lambda h = S_F h, \qquad S_F\, h = F \circ (\operatorname{id} + h), \quad \operatorname{ad}_\Lambda = [\Lambda, \cdot\,]. \tag{5.7}$$

We will show in an instant that the operator $\operatorname{ad}_\Lambda^{-1} \circ S_F$ restricted on the space $\mathcal{B}_\rho$ has a fixed point h, if $\rho > 0$ is sufficiently small,

$$h = \bigl(\operatorname{ad}_\Lambda^{-1} \circ S_F\bigr)(h), \qquad h \in \mathcal{B}_\rho. \tag{5.8}$$

Applying to both parts the operator $\operatorname{ad}_\Lambda$, we conclude that h solves (5.7) and therefore $\operatorname{id} + h$ conjugates the linear field Λx with the nonlinear field $\Lambda x + F(x)$ in the polydisk $\{|x| < \rho\}$.

Consider this operator $\operatorname{ad}_\Lambda^{-1} \circ S_F$ in the space $\mathcal{B}_\rho$ with sufficiently small ρ. The operator $\operatorname{ad}_\Lambda^{-1}$ is bounded by Lemma 5.11; its norm is the reciprocal to the smallest small divisor and is independent of ρ. On the other hand, the argument shift operator S_F is strongly contracting with the contraction rate (Lipschitz constant) going to zero with ρ as $O(\rho)$. Thus the composition will be contracting on the ρ-ball B_ρ in the ρ-majorant norm with the contraction rate $O(1) \cdot O(\rho) = O(\rho) \to 0$. By the contracting map principle, there exists a unique fixed point of the operator equation (5.8) in the space $\mathcal{B}_\rho$

which is therefore a holomorphic vector function. The corresponding map $H = (\mathrm{id} + h)^{-1}$ linearizes the holomorphic vector field. $\square$

5C. Resonant case: polynomial normal form. Modification of the previous construction allows us to prove that a resonant holomorphic vector field in the Poincaré domain can be brought into a *polynomial* normal form.

Consider a holomorphic vector field $F(x) = Ax + V(x)$ with the linearization matrix A having eigenvalues in the Poincaré domain, and nonlinear part V of order $\geqslant 2$ (i.e., 1-flat) at the origin. Without loss of generality (passing, if necessary, to an orbitally equivalent field cF, $0 \neq c \in \mathbb{C}$), one may assume that the eigenvalues of A satisfy the condition

$$1 < \operatorname{Re} \lambda_j < r \qquad \forall j = 1, \dots, n \tag{5.9}$$

with some natural $r \in \mathbb{N}$.

Theorem 5.15 (A. M. Lyapunov, H. Dulac)**.** *If the eigenvalues of the linearization matrix A of a holomorphic vector field $F(x) = Ax + V(x)$ satisfy the condition* (5.9) *with some integer $r \in \mathbb{N}$, then the holomorphic vector field $F(x)$ is locally holomorphically equivalent to any holomorphic vector field with the same r-jet.*

Proof. A holomorphic conjugacy $H = \mathrm{id} + h$ between the fields F and $F + g$ satisfies the functional equation $\left(\frac{\partial H}{\partial x}\right) F = (F + g) \circ H$ which can be expanded to

$$\left(\frac{\partial h}{\partial x}\right) Ax - Ah = (V \circ (\mathrm{id} + h) - V) + g \circ (\mathrm{id} + h) - \left(\frac{\partial h}{\partial x}\right) V. \tag{5.10}$$

Consider the three operators,

$$T_V \colon h \mapsto V \circ (\mathrm{id} + h) - V, \quad S_g \colon h \mapsto g \circ (\mathrm{id} + h), \qquad \Psi \colon h \mapsto \left(\frac{\partial h}{\partial x}\right) V.$$

Using these three operators, the differential equation (5.10) can be written in the form

$$\mathrm{ad}_A\, h = Th + Sh + \Psi h, \tag{5.11}$$

where $T = T_V$, $S = S_g$ and, as before in (5.7), ad_A is the commutator with the *linear* field $\mathbf{A}(x) = Ax$. The key difference with the previous case is two-fold: first, because of the resonances, the operator ad_A is *not invertible* anymore, and second, since the field F is nonlinear, the additional operator Ψ occurs in the right hand side. Note that this operator is a derivation of h, thus is unbounded in *any* majorant norm $[\![\cdot]\!]_\rho$.

Let $\mathcal{B}_{m,\rho} = \{f \colon j^m f = 0\} \cap \mathcal{B}_\rho$ be a subspace of m-flat series in the Banach space $\mathcal{B}_\rho$, equipped with the same majorant norm $[\![\cdot]\!]_\rho$. Since V is 1-flat, all three operators T, S, Ψ map the subspace $\mathcal{B}_{m,\rho}$ into itself for any $m > 1$.

Moreover, by Lemma 5.14, the argument shift operator S is strongly contracting, regardless of the choice of m. The “finite difference” operator T_V differs from the argument shift, S_V by the constant operator $V = T(0)$ which does not affect the Lipschitz constant. Since $\|\!| V |\!\|_\rho = O(\rho^2)$, the operator T is also strongly contracting.

The operator ad_A preserves the order of all monomial terms and hence also maps $\mathcal{B}_{m,\rho}$ into itself for all m, ρ, and is *invertible* on these spaces if m is sufficiently large. Indeed, if $|\alpha| > r+1$, then by (5.9) $\operatorname{Re}(\langle\alpha,\lambda\rangle - \lambda_j) > 0$, and all denominators in the formula

$$\operatorname{ad}_A^{-1}|_{\mathcal{B}_{m,\rho}} : \sum_{|\alpha|\geqslant m} c_{k\alpha}\, x^\alpha \frac{\partial}{\partial x_j} \longmapsto \sum_{|\alpha|\geqslant m} \frac{c_{k\alpha}}{\langle\alpha,\lambda\rangle - \lambda_j}\, x^\alpha \frac{\partial}{\partial x_j} \tag{5.12}$$

are nonzero if $m \geqslant r+1$, and the restriction of ad_A^{-1} on $\mathcal{B}_{m,\rho}$ is bounded. Moreover,

$$\|\!| \operatorname{ad}_A^{-1} h |\!\|_\rho \leqslant O(1/m)\, \|\!| h |\!\|_\rho \tag{5.13}$$

uniformly over all $h \in \mathcal{B}_{m,\rho}$ of order $m \geqslant r+1$.

Thus the two compositions, $\operatorname{ad}_A^{-1} \circ S$ and $\operatorname{ad}_A^{-1} \circ T$, are strongly contracting. To prove the theorem, it remains to prove that the *linear* operator $\operatorname{ad}_A^{-1} \circ \Psi \colon \mathcal{B}_{m,\rho} \to \mathcal{B}_{m,\rho}$ is strongly contracting when m is larger than $r+1$.

Consider the $\|\!| \cdot |\!\|_\rho$-normalized vectors $h_{k\beta} = \rho^{-|\beta|} x^\beta \frac{\partial}{\partial x_k}$ for all $k = 1, \dots, m$ and all $|\beta| \geqslant m$ spanning the entire space $\mathcal{B}_{m,\rho}$. We prove that

$$\|\!| \operatorname{ad}_A^{-1} \Psi h_{k\beta} |\!\|_\rho = O(\rho) \qquad \text{as } \rho \to 0 \tag{5.14}$$

uniformly over $|\beta| \geqslant m$ and all k. Since $\operatorname{ad}_A^{-1} \circ \Psi$ is linear, this would imply that $\operatorname{ad}_A^{-1} \circ \Psi$ is strongly contracting.

The direct computation yields

$$\Psi h_{k\beta} = \sum_{i=1}^{n} \rho^{-|\beta|} \frac{\beta_i}{x_i}\, x^\beta\, V_i \frac{\partial}{\partial x_k}.$$

Since V is 1-flat, $\|\!| V_i |\!\|_\rho = O(\rho^2)$; substituting this into the definition of the majorant norm, we obtain

$$\|\!| \Psi h_{k\beta} |\!\|_\rho \leqslant \sum_i \beta_i\, \rho^{-1} O(\rho^2) = \beta_i\, O(\rho),$$

where $O(\rho)$ is uniform over all β. Since the order of the products $\frac{x^\beta}{x_i} V_i$ is at least $|\beta|+1$, by (5.13) we have

$$\|\!| \operatorname{ad}_A^{-1} \Psi h_{k\beta} |\!\|_\rho \leqslant \frac{\beta_i}{|\beta|}\, O(\rho) = O(\rho)$$

uniformly over all β with $|\beta| \geqslant m \geqslant r+1$. Thus the last remaining composition $\operatorname{ad}_A^{-1} \circ \Psi$ is also strongly contracting, which implies existence of a solution for the fixed point equation

$$h = \operatorname{ad}^{-1} \circ (T + S + \Psi)h$$

equivalent to (5.11), in a sufficiently small polydisk $\{|x| < \rho\}$. □

Now one can easily complete the proof of the holomorphic normalization theorem in the Poincaré domain in the resonant case.

Proof of Theorem 5.5 (resonant case). By the Poincaré–Dulac normalization process, one can eliminate all nonresonant terms up to any finite order m by a polynomial transformation. By Theorem 5.15, m-flat holomorphic terms can be eliminated by a holomorphic transformation if m is large enough (depending on the spectrum of the linearization matrix). □

Remark 5.16. In the Poincaré domain one can prove an even stronger claim: if a holomorphic vector field depends analytically on finitely many additional parameters $\lambda \in (\mathbb{C}^m, 0)$ and belongs to the Poincaré domain for $\lambda = 0$, then by a holomorphic change of variables holomorphically depending on parameters, the field can be brought to a polynomial normal form involving only resonant terms. In such a form this assertion is formulated in [**Bru71**]. The proof can be achieved by minor adjustment of the arguments used in the demonstration of Theorem 5.15.

5D. Holomorphic normal forms for self-maps. In the same way as the formal theory for vector fields $\mathcal{D}[[\mathbb{C}^n, 0]]$ and maps $\operatorname{Diff}[[\mathbb{C}^n, 0]]$ are largely parallel (see §4**G**), the analytic theory of vector fields and biholomorphisms are also parallel.

The additive resonance conditions $\lambda_j - \langle \alpha, \lambda \rangle \neq 0$ correspond to the multiplicative resonance conditions $\mu_j^{-1}\mu^\alpha \neq 1$. The additive Poincaré condition (Definition 5.1) requires that (eventually after a rotation) all eigenvalues λ_j of the vector field lie to one side of the imaginary axis. Its multiplicative counterpart requires that all eigenvalues μ_j of the map must be to one side of the unit circle. Such maps are automatically contracting or expanding, and admit at most finitely many multiplicative resonance relations between the eigenvalues.

The result parallel to the Poincaré Theorem 5.5 takes the following form. Let $M \in \mathrm{GL}(n, \mathbb{C})$ be a matrix in the upper triangular Jordan normal form with the eigenvalues $\mu_1, \dots, \mu_n \in \mathbb{C}^*$. The Poincaré–Dulac normal form is a map

$$f \colon \mathbb{C}^n \to \mathbb{C}^n, \qquad x \mapsto f(x) = Mx + \sum_{\substack{\alpha \in \mathbb{Z}_+,\ |\alpha| \geqslant 2 \\ \mu_j = \mu^\alpha}} x^\alpha \mathbf{e}_j, \tag{5.15}$$

where $\mathbf{e}_j \in \mathbb{C}^n$ is the jth basis vector. If M is in the *multiplicative Poincaré domain*, i.e., if the eigenvalues are all of modulus less than one or all of modulus greater than one, then the normal form (5.15) is polynomial (contains finitely many terms).

The general result for holomorphic self-maps in the Poincaré domain has the following form.

Theorem 5.17. *A holomorphic invertible map $f \in \operatorname{Diff}(\mathbb{C}^n, 0)$ with the spectrum $\mu_1, \dots, \mu_n$ inside the unit disk, $0 < |\mu_j| < 1$, $j = 1, \dots, n$, is analytically equivalent to its polynomial Poincaré–Dulac formal normal form* (5.15). □

In the important particular case of one-dimensional maps, the multiplicative Poincaré condition holds automatically if the map is *hyperbolic*, i.e., if its multiplicator μ has modulus different from one. This automatically guarantees that resonances are impossible, and hence the Poincaré–Dulac normal form (5.15) is linear. The corresponding result was proved by E. Schröder (1870) and A. Kœnigs (1884).

Theorem 5.18. *A holomorphic germ $f\colon (\mathbb{C}, 0) \to (\mathbb{C}, 0)$, $f(x) = \mu x + O(x^2)$, is analytically linearizable if $|\mu| \neq 1$.*

If $f = f_t$ depends analytically on additional parameter $t \in U \subseteq \mathbb{C}^p$, the linearizing chart can also be chosen analytically depending on this parameter as soon as the respective multiplier μ_t remains off the unit circle.

Because of its importance, we will give an independent proof of this result by the path method in §5**F** below. Yet another (shortest known) proof is outlined in Problem 5.6.

5E. Linearization in the Siegel domain: Siegel, Brjuno and Yoccoz theorems (micro-survey). In the Siegel domain the denominators $\lambda_j - \langle \alpha, \lambda \rangle$ are not separated from zero, hence even in the nonresonant case the operator $\operatorname{ad}_A = [\mathbf{A}, \cdot\,]$ of commutation with the linear part of the field has unbounded inverse ad_a^{-1}. Yet since the operator S_F is *strongly contracting*, the equation (5.7) can be solved with respect to h by Newton-type iterations, provided that the small denominators $|\lambda_j - \langle \alpha, \lambda \rangle\,|$ do not approach zero too fast as $|\alpha| \to \infty$.

The corresponding technique is known under the general name of *KAM theory* (after A. Kolmogorov, V. Arnold and J. Moser). The issue is very classical; accurate formulations and proofs can be found in many excellent sources, e.g., [**CG93, Arn83**]. We formulate only the basic results.

Definition 5.19. A tuple of complex numbers $\lambda \in \mathbb{C}^n$ from the Siegel domain $\mathfrak{S}$ is called *Diophantine*, if the small denominators decay no faster than polynomially with $|\alpha|$, i.e.,

$$\exists C, N < +\infty \text{ such that } \quad \forall \alpha \in \mathbb{Z}_+^n, \quad |\lambda_j - \langle \alpha, \lambda \rangle\,|^{-1} \leqslant C\,|\alpha|^N. \qquad (5.16)$$

Otherwise the tuple (vector, collection) is called *Liouvillean.*

Liouvillean vectors are scarce: the set of points $\lambda \in \mathbb{C}^n$ satisfying violating the condition 5.16 with a given N, has Lebesgue measure zero in $\mathfrak{S} \subset \mathbb{C}^n$ if $N > (n-2)/2$; see [**Arn83**].

Theorem 5.20 (Siegel theorem)**.** *If the linearization matrix $\varLambda$ of a holomorphic vector field is nonresonant of Siegel type and has Diophantine spectrum, then the field is holomorphically linearizable.*

Thus the majority (in the sense of Lebesgue measure) of germs of holomorphic vector fields are analytically linearizable. Yet one may further relax sufficient conditions for convergence of linearizing series in the Siegel domain.

Definition 5.21. A nonresonant collection $\lambda \in \mathbb{C}^n$ is said to satisfy the *Brjuno condition*, if the small denominators decrease *sub-exponentially*,

$$|\lambda_j - \langle \alpha, \lambda \rangle\,|^{-1} \leqslant C e^{|\alpha|^{1-\varepsilon}}, \qquad \text{as } |\alpha| \to \infty, \qquad (5.17)$$

for some finite C and positive $\varepsilon > 0$.

Theorem 5.22 (Brjuno theorem)**.** *A holomorphic vector field with nonresonant linearization matrix of Siegel type satisfying the Brjuno condition, is holomorphically linearizable.*

On the other hand, if the denominators $|\lambda_j - \langle \alpha, \lambda \rangle\,|$ accumulate to zero too fast, e.g., *super-exponentially*, then the corresponding germs are in general nonlinearizable (cf. with Remark 5.33 below).

Analogs of the Siegel and Brjuno theorems hold for holomorphic germs. The most important case is that of one-dimensional *conformal germs* from the group $\operatorname{Diff}(\mathbb{C}^1, 0)$. Such germs belong to the Siegel domain if and only if their multiplicator μ belongs to the unit circle, $\mu = \exp 2\pi i l$, with some $l \in \mathbb{R}$; they are nonresonant if l is an irrational number. The Diophantine and Brjuno conditions translate for this case as assumptions that this irrational number $l \in \mathbb{R} \smallsetminus \mathbb{Q}$ does not admit abnormally accurate rational approximations.

For instance, if the complex number $\mu = \exp 2\pi i l$, $l \in \mathbb{R}$, satisfies the *multiplicative Brjuno condition*

$$|\mu^k - 1|^{-1} < C e^{k^{1-\varepsilon}}, \qquad C < +\infty,\ \varepsilon > 0, \qquad (5.18)$$

then any holomorphic map $(\mathbb{C}, 0) \to (\mathbb{C}, 0)$, $z \mapsto \mu z + z^2 + \cdots$, is holomorphically linearizable. The sufficient arithmetic condition (5.18) turns out to also be *necessary* in the following sense.

Theorem 5.23 (J.-C. Yoccoz [**Yoc88, Yoc95**]). *If the complex number* $\mu = \exp 2\pi i l$, $l \in \mathbb{R}$, *violates the multiplicative Brjuno condition* (5.18), *then there exists a nonlinearizable holomorphic germ* $(\mathbb{C}, 0) \to (\mathbb{C}, 0)$, $z \mapsto \mu z + f(z)$, $f(z) = z^2 + \cdots$.

In fact, in the assumptions of this theorem for almost all complex numbers $w \in \mathbb{C}$ the germ $f_w(z) = \mu z + w f(z)$ is analytically nonlinearizable; cf. with Theorem 5.29 below and [**PM01**].

Remark 5.24. The condition on the rate of convergence of small denominators can be reformulated in terms of the growth rate of coefficients of decomposition of the irrational number $l \in \mathbb{R} \smallsetminus \mathbb{Q}$ into the continuous fraction. This is a more standard way of formulating the Brjuno condition in the recent literature.

If a resonance occurs in the Siegel case, then the situation turns out to be even more complicated: a resonant conformal germ $f \in \operatorname{Diff}(\mathbb{C}, 0)$ with multiplicator $\mu \in \exp 2\pi i \mathbb{Q}$ is almost never analytically equivalent to its polynomial Poincaré–Dulac formal normal form described in Theorem 4.26. This result and its numerous developments are explained in detail in §21.

A two-dimensional analytic orbital classification of Siegel resonant vector fields (saddle-nodes and resonant saddles from Table I.1) is at least as difficult as the analytic classification of resonant germs from $\operatorname{Diff}(\mathbb{C}, 0)$. Indeed, in §7 we will show that the corresponding foliations have leaves with nontrivial (infinite cyclic) fundamental group, whose holonomy is generated by Siegel resonant germs from $\operatorname{Diff}(\mathbb{C}, 0)$. The details can be found in Chapter IV; see §22.

Somewhat unexpectedly, the cuspidal points behave better than their less degenerate brethren. In [**SZ02**] H. Żołądek and E. Stróżyna proved that one can always reduce a holomorphic planar vector field near a cuspidal singular point to a *holomorphic* normal form (4.17) (i.e., with converging series $a(x), b(x) \in \mathcal{O}(\mathbb{C}, 0)$) by a biholomorphic transformation. The direct and difficult proof from [**SZ02**] was recently replaced by beautiful geometric arguments by F. Loray [**Lor06**]. This proof, based on *nonlocal uniformization technique*, is split into a series of problems in §23 (Problems 23.6–23.13).

5F. Path method. In this section we outline another very powerful *analytic* method of reducing holomorphic vector fields and self-maps to their normal forms. This method is called *path method* (méthode de chemin, homotopy method) since it consists of connecting the initial object (field, self-map) with its normal form by a path (usually a line segment) and then looking for a flow of a nonautonomous vector field that would conjugate with each other all objects in this parametric family.

We *illustrate* the path method by proving two relatively simple results, analytic reducibility of one-dimensional holomorphic vector fields (cf. with Theorem 4.24) and one-dimensional hyperbolic self-maps to their normal form. Both results, however, can be proved by shorter arguments; see Problems 5.5 and 5.6 below.

Theorem 5.25. *Any analytic vector field* $F(x) = x^{k+1}(1+\cdots)\frac{\partial}{\partial x} \in \mathcal{D}(\mathbb{C},0)$ *is analytically conjugate to its polynomial formal normal form* $F_0(x) = (x^{k+1} + ax^{2k+1})\frac{\partial}{\partial x}$.

Proof. Without loss of generality we may assume from the beginning, that the jet of F of any specified order is already reduced to the normal form. Thus we can assume that the field $F = F_1$ is given as $F_0(x) + R(x)\frac{\partial}{\partial x}$, where R is as flat at the origin, as necessary. It will be sufficient to require that the function $R(x)$ has zero of multiplicity $2k+2$ at the origin, $R(x) = x^{2k+2}S(x)$, $S \in \mathcal{O}(\mathbb{C},0)$. We want to show that for all values of an auxiliary complex parameter z from some domain $U \subseteq \mathbb{C}$ containing the segment $[0,1]$, the vector fields $F_z(x) = F_0(x) + zR(x)\frac{\partial}{\partial x}$ are holomorphically equivalent to each other. This, in particular, would imply that F_0 and F_1 are holomorphically equivalent, which would immediately imply the assertion of the theorem.

Consider the planar domain $(\mathbb{C},0)\times U$ and the vector field on it,

$$\mathbf{F} = \mathbf{F}_0 + zR(x)\cdot\tfrac{\partial}{\partial x} + 0\cdot\tfrac{\partial}{\partial z}, \qquad \mathbf{F}_0 = (x^{k+1} + ax^{2k+1})\tfrac{\partial}{\partial x}, \tag{5.19}$$

which is the suspension of the above parametric family of vector fields on the line.

Consider another planar vector field $\mathbf{H} \in \operatorname{Diff}\big((\mathbb{C}^1,0)\times U\big)$, $U \subseteq \mathbb{C}$, which has the form

$$\mathbf{H} = h(x,z)\tfrac{\partial}{\partial x} + 1\cdot\tfrac{\partial}{\partial z}, \qquad h(0,z) \equiv 0. \tag{5.20}$$

Lemma 5.26 (Path method paradigm). *If there exists a holomorphic vector field* $\mathbf{H} \in \operatorname{Diff}\big((\mathbb{C}^1,0)\times U\big)$ *of the form* (5.20) *which* commutes *with* $\mathbf{F}$*,*

$$[\mathbf{F},\mathbf{H}] = 0, \tag{5.21}$$

then all germs of vector fields $F_z \in \mathcal{D}(\mathbb{C}^1,0)$ *are holomorphically equivalent to each other for all values of* $z \in U$*.*

Proof. If the vector fields $\mathbf{F}$ and $\mathbf{H}$ commute, then the flow of the vector field $\mathbf{H}$ commutes with the flow of $\mathbf{F}$ and hence the flow maps of $\mathbf{H}$ are symmetries of the field $\mathbf{F}$.

Because of the special structure of $\mathbf{H}$, its flow sends the lines $\{z = \text{const}\}$ into each other, each time fixing the origin $\{x = 0\}$. Thus the flow $\exp\mathbf{H}$ maps $\{z = 0\}$ into $\{z = 1\}$, is defined in some neighborhood of the origin and conjugating $\mathbf{F}|_{z=0} = F_0$ with $\mathbf{F}|_{z=1} = F_1$. □

Now we can complete the proof of Theorem 5.25, showing that in the assumptions of the theorem, such a vector field $\mathbf{H}$ indeed exists. The *homological equation* (5.21) is equivalent to a partial differential equation on the function H,

$$f\cdot\frac{\partial h}{\partial x} - h\cdot\frac{\partial f}{\partial x} = -R, \qquad f(x,z) = x^{k+1} + ax^{2k+1} + zR(x). \tag{5.22}$$

Yet in fact this equation can be considered as a linear first order ordinary (with respect to the x-variable) nonhomogeneous differential equation analytically depending on the parameter $z \in U$. The solution of the corresponding *homogeneous* equation is immediate,

$h_0(x,z) = f(x,z)$, using the ansatz $h(x,z) = s(x,z)h_0(x,z)$ we obtain the equation (recall that $R = x^{2k+2}S$),

$$f^2 \cdot \frac{\partial s}{\partial z} = -R(x), \quad \text{i.e.} \quad \frac{\partial s}{\partial x} = -\frac{S(x)}{\bigl(1 + ax^k + x^{k+1}S(x)\bigr)^2}. \tag{5.23}$$

Integration of the right hand side with the initial condition $s(0,z) = 1$ yields a solution $s = s(x,z)$ holomorphic at $x = 0$ for all $z \in U$. The vector field $\mathbf{H} = s(x,z)\mathbf{F} + 1 \cdot \frac{\partial}{\partial z}$ satisfies all conditions imposed by Lemma 5.26 and allows us to construct a holomorphic conjugacy between F_0 and F_1. □

Obviously, the polynomial normal form (4.21) can be replaced by the rational normal form (4.23).

Remark 5.27. Besides holomorphic differential vector fields, one may consider *meromorphic differential* 1*-forms* on the complex line (or, more precisely, their germs at the origin): the set of all such forms is naturally denoted by $\Lambda^1(\mathbb{C},0) \otimes \mathfrak{M}(\mathbb{C},0)$.

The group $\operatorname{Diff}(\mathbb{C},0)$ acts on such forms, so one can establish normal forms. Yet instead of developing parallel theory, one can use duality: a 1-form $\omega \in \Lambda^1(\mathbb{C},0)$ and a vector field $F \in \mathcal{D}(\mathbb{C},0)$ are called *dual*, if $\omega(F) \equiv 1$. Holomorphic transform of a dual pair is again a dual pair.

Meromorphic (i.e., with a pole at the origin) 1-forms have two obvious invariants that cannot be changed by holomorphic transformations: the *order of the pole* and the *residue* at this point.

The form dual to the rational vector field (4.23) is $\frac{dx}{x^{k+1}} - a\frac{dx}{x}$, and the formal invariant $a \in \mathbb{C}$ is the residue of this form (modulo the sign). This observation explains the role of the formal invariant.

As yet another application of the path method, we give an independent proof of the Schröder–Kœnigs Theorem 5.18.

Consider the analytic self-map $f \in \operatorname{Diff}(\mathbb{C},0)$, $f(x) = \mu x + r(x)$, with the multiplicator $\mu \in \mathbb{C}^*$, $|\mu| < 1$ and analytic nonlinearity $r(x) = O(x^2)$.

As before, we embed f into an analytic one-parameter deformation $f_z(x) = \mu x + zr(x)$ with a complex parameter $z \in U \subseteq \mathbb{C}$, $[0,1] \subseteq U$, and suspend it to the planar self-map $\mathbf{f} \in \operatorname{Diff}(\mathbb{C}^2,0)$,

$$\mathbf{f}\colon (x,z) \mapsto (\mu x + zr(x), z), \qquad (x,z) \in (\mathbb{C}^1,0) \times U. \tag{5.24}$$

The following lemma is a reformulation of the main paradigm of the path method (Lemma 5.26) for the current context.

Lemma 5.28. *If a vector field* $\mathbf{H}$ *as in* (5.20) *is preserved by the self-map* $\mathbf{f}$*, i.e.,*

$$\mathbf{f}_* \cdot \mathbf{H} = \mathbf{H} \circ \mathbf{f}, \qquad \mathbf{f}_* = \frac{\partial \mathbf{f}(x,z)}{\partial(x,z)}, \tag{5.25}$$

then all self-maps f_z *for all* $z \in U$*, are analytically equivalent, in particular,* $f_1 = f$ *is analytically equivalent to the linear map* f_0*. The conjugacy is achieved by the flow of the field* $\mathbf{H}$ *restricted on the lines* $\{z = \text{const}\}$. □

The proof of Lemma 5.28 almost literally reproduces that of Lemma 5.26 and is skipped. In order to prove Theorem 5.18, we need only to show that the homological equation (5.25) is solvable.

Alternative proof of Theorem 5.18. The identity (5.25) reduces to a single scalar linear nonhomogeneous functional equation

$$\frac{\partial f_z(x)}{\partial x}\cdot h(x,z)-h(f_z(x),z)=r(x). \tag{5.26}$$

This equation can be solved in two steps, solving first the corresponding homogeneous equation $(\frac{\partial f}{\partial x})u-u\circ f=0$, and then looking for a solution of (5.26) in the form $h=su$, similar to the way the equation (5.22) was solved.

The homogeneous equation can be rewritten as a fixed point statement,

$$h=\left(\frac{\partial f}{\partial x}\right)^{-1}\cdot(h\circ f),\qquad f=f_z\in\operatorname{Diff}(\mathbb{C},0). \tag{5.27}$$

It has a trivial (zero) solution, yet we can restrict the operator occurring in the right hand side, on the subspace of functions tangent to identity, $h(x)=x+O(x^2)$.

Without loss of generality we may assume (passing to a sufficiently small neighborhood of the origin which is rescaled to the unit disk) that all maps f_z satisfy the inequalities

$$\begin{gathered}\left|\frac{\partial f}{\partial z}\right|\geqslant\mu_-,\qquad |f(x)|<\mu_+|x|,\qquad \forall x\in D_1=\{|x|\leqslant 1\},\\ 0<\mu_-<|\mu|<\mu_+<1.\end{gathered} \tag{5.28}$$

Here $\mu_\pm$ are two positive constants which can be assumed to be arbitrarily close to $|\mu|<1$.

First we show that the operator $\varPhi\colon h\mapsto(\frac{\partial f}{\partial x})^{-1}\cdot(h\circ f)$ restricted on the subspace

$$\mathcal{M}=\{u\in\mathcal{A}(D_1)\colon u(0)=0,\ \tfrac{du}{dx}=1\}$$

of holomorphic functions tangent to the identity at the origin, is contracting in the sense of the usual supremum-norm $\|u\|=\max_{x\in D_1}|u(x)|$. Clearly, $\varPhi(\mathcal{M})\subseteq\mathcal{M}$.

Indeed, since $\varPhi$ is linear, it is sufficient to show that $\|\varPhi q\|<\lambda\|q\|$ for any $q\in\mathcal{A}(D_1)$ having a second order zero at the origin and some λ strictly between 0 and 1. Note that for any such function $q(x)$, we have the inequality $|q(x)|\leqslant\|q\|\cdot|x|^2$: it is sufficient to apply the maximum modulus principle to the holomorphic ratio $q(x)/x^2$. Then from (5.28) it immediately follows that

$$\|\varPhi q\|\leqslant\max_{|x|\leqslant 1}\frac{1}{\mu_-}\|q\|\cdot|f(x)|^2\leqslant\frac{\mu_+^2}{\mu_-}\cdot\max_{|x|\leqslant 1}\|q\|\,|x|^2\leqslant\frac{\mu_+^2}{\mu_-}\cdot\|q\|.$$

Since the ratio μ_+^2/μ_- can be made arbitrarily close to $|\mu|<1$, the operator $\varPhi$ restricted on $\mathcal{M}$ is contracting and hence has a holomorphic fixed point u analytically depending on z and any additional parameters (if present).

Now a solution of the nonhomogeneous equation can be found using the ansatz $h=su$. Substituting this ansatz into the equation (5.26), we obtain the *Abel-type equation*

$$s-s\circ f=-R(x),\qquad R=R_z(x)=\frac{r(x)}{(\frac{\partial f}{\partial z})\cdot u(x,z)},\qquad f=f_z(x). \tag{5.29}$$

The function $R_z(x)$ is holomorphic and vanishes at the origin $x=0$ for all values of x, since f_z has a simple zero and $r(x)$ has a double zero at the origin.

The formal solution of the equation (5.29) is given by the series

$$s=-\sum_{k=0}^{\infty}R\circ f^{\circ k},\qquad s=s(\cdot,z),\ f=f_z,\ R=R_z, \tag{5.30}$$

which is well defined because f is contracting. Moreover, since R vanishes at the origin, we have $|R_z(x)| < C|x|$ for some $C < \infty$ and all $x \in D_1$. Combining this with the uniform bounds $|f^{\circ k}(x)| \leqslant \mu_+^k |x|$ implied by (5.28), we conclude that the series (5.30) converges uniformly on D_1 and hence its sum is a holomorphic function vanishing at $x = 0$. The holomorphic vector field $\mathbf{H} = s(x,z)u(x,z)\frac{\partial}{\partial x} + 1 \cdot \frac{\partial}{\partial z}$ solves the equation (5.25).

The alternative proof of Theorem 5.18 is complete. ☐

* * *

5G. Divergence dichotomy. As follows from the Poincaré, Siegel and Brjuno theorems, for most linear parts the linearizing series converges, and in the remaining cases the linearizing series *may diverge*. On the other hand, no matter how "bad" the linearization and its eigenvalues are, there are always nonlinear systems that can be linearized (e.g., linear systems in nonlinear coordinates). It turns out that in some precise sense for a given linear part, the convergence/divergence pattern is common for *most* nonlinearities.

Consider a *parametric* nonlinear system

$$\dot x = Ax + z\,f(x), \qquad x \in \mathbb{C}^n, \quad z \in \mathbb{C}, \tag{5.31}$$

holomorphic in some neighborhood of the origin with the *nonresonant* linearization matrix A and the nonlinear part linearly depending on the auxiliary complex parameter $z \in \mathbb{C}$. For such systems for each value of the parameter $z \in \mathbb{C}$ there is a *unique* (by Remark 4.6) formal series $H_z(x) = x + h_z(x) \in \operatorname{Diff}[[x,z]]$ linearizing (5.31). This series may converge for some values of z while diverging for the rest. It turns out that there is a strict alternative: either the linearizing series *converges for all values of z without exception*, or on the contrary *the series H_z diverges for all z outside a rather small exceptional set $K \Subset \mathbb{C}$.*

The exceptional sets are small in the sense that their (electrostatic) *capacity* is zero. The notion of capacity is formally introduced below in §5**H**, where some of its basic properties are collected. We mention here only that zero capacity implies zero Lebesgue measure for any compact set.

Theorem 5.29 (Divergence dichotomy, Yu. Ilyashenko [**Ily79a**], R. Perez Marco [**PM01**])**.** *For any nonresonant linear family* (5.31) *one has the following alternative*:

1. *Either the linearizing series $H_z \in \operatorname{Diff}[[\mathbb{C}^n,0]]$ converges for all values of $z \in \mathbb{C}$ in a symmetric polydisk $\{|x| < r\}$ of a positive radius $r = r(z) > 0$ decreasing as $O(|z|^{-1})$ as $z \to \infty$, or*
2. *The linearizing series H_z diverges for all values of z except for a set $K_f \Subset \mathbb{C}$ of capacity zero.*

The proof is based on the following property of polynomials, which can be considered as a quantitative uniqueness theorem for polynomials. If K is a set of positive capacity and $p \in \mathbb{C}[z]$ a polynomial vanishing on K, then by definition p vanishes identically. One can expect that if p is small on K, then it is also uniformly small on *any* other compact subset, in particular, on all compact subsets of $\mathbb{C}$.

Theorem 5.30 (Bernstein inequality). *If $K \Subset \mathbb{C}$ is a set of positive capacity, then for any polynomial $p \in \mathbb{C}[z]$ of degree $r \geqslant 0$,*

$$|p(z)| \leqslant \|p\|_K \exp(rG_K(z)), \tag{5.32}$$

where $\|p\|_K = \max_{z\in K} |p(z)|$ is the supremum-norm of p on K, and $G_K(z)$ is the nonnegative Green function of the complement $\mathbb{C} \smallsetminus K$ with the source at infinity; see (5.36).

We postpone the proof of this theorem until §5**H** and proceed with deriving Theorem 5.29 from the Bernstein inequality.

Lemma 5.31. *Formal Taylor coefficients of the formal series linearizing the field* (5.31) *are polynomial in z.*

More precisely, every monomial x^α, $|\alpha| \geqslant 2$, enters into the vector series h_z with the coefficient which is a polynomial of degree $\leqslant |\alpha| - 1$ in z.

Proof. The equation determining $h = h_z$ is of the form

$$\left(\frac{\partial h_z}{\partial x}\right)(Ax + z\,f(x)) = Ah_z(x). \tag{5.33}$$

Collecting the terms of degree m in x, we obtain for the corresponding mth homogeneous (vector) components $h_z^{(m)}$, $f^{(l)}$, the recurrent identities

$$\left(\frac{\partial h_z^{(m)}}{\partial x}\right) Ax - Ah_z^{(m)} = -z \sum_{k+l=m,\, l\geqslant 2} \left(\frac{\partial h_z^{(k+1)}}{\partial x}\right) f^{(l)}.$$

From these identities it obviously follows by induction that each $h_z^{(m)}$ is a polynomial of degree $m-1$ in z for all $m \geqslant 1$ (recall that f does not depend on z). $\square$

Proof of Theorem 5.29. Assume that the formal series $H_z(x) = x+h_z(x)$ linearizing the field $F_z(x) = Ax + z\,f(x)$ converges for values of z belonging to some set $K^* \subset \mathbb{C}$ of positive capacity.

Consider the subsets $K_{c\rho} \Subset \mathbb{C}$, $\rho > 0$, $c < +\infty$, defined by the condition

$$z \in K_{c\rho} \iff |h_z^{(m)}(0)| \leqslant c\rho^{-m} \quad \forall m \in \mathbb{N}.$$

By this definition, $K^* = \bigcup_{c,\rho} K_{c\rho}$, since a Taylor series converges if and only if satisfies some Cauchy-type estimate. Each of the sets $K_{c\rho}$ obviously is a compact subset of $\mathbb{C}$, being an intersection of semialgebraic compact sets.

The compacts $K_{c\rho}$ are naturally nested: $K_{c'\rho'} \subseteq K_{c\rho}$ if $\rho' > \rho$ and $c' < c$. Passing to a countable sub-collection, one concludes that the set K of positive capacity is a countable union of compacts $K_{c\rho}$. By Proposition 5.35 (see below), one of these compacts must also be of positive capacity. Denote this compact by $K = K_{c\rho}$; by its definition,

$$|h_z^{(m)}| \leqslant c\rho^{-m}, \qquad \forall z \in K,\ \forall m \in \mathbb{N}.$$

Since the capacity of K is positive, Theorem 5.30 applies. By this theorem and Lemma 5.31, the polynomial coefficients of the series h_z for *any* $z \in \mathbb{C}$ satisfy the inequalities

$$|h_z^{(m)}| \leqslant c\rho^{-m}\exp[(m-1)G_K(z)] \leqslant c(\rho/\exp G_K(z))^{-m}, \quad \forall z \in \mathbb{C},\ \forall m \in \mathbb{N}.$$

This means that the series h_z converges for any $z \in \mathbb{C}$ in the symmetric polydisk $\{|x| < \rho/\exp G_K(z)\}$. Together with the asymptotic growth rate $G_K(z) \sim \ln|z| + O(1)$ as $z \to \infty$ (see (5.36)) this proves the lower bound on the convergence radius of H_z. $\square$

The dichotomy established in Theorem 5.29 may be instrumental in constructing "nonconstructive" examples of diverging linearization series. Consider again the nonresonant case where the homological equation $\operatorname{ad}_A g = f$ is always formally solvable.

Theorem 5.32 ([**Ily79a**]). *Assume that the formal solution $g \in \mathcal{D}[[\mathbb{C}^n, 0]]$ of the homological equation $\operatorname{ad}_A g = f$ is divergent.*

Then the series linearizing the vector field $F_z(x) = Ax + z\,f(x)$, diverges for most values of the parameter z, eventually except for a zero capacity set.

Proof. Assume the contrary, that the linearizing series H_z converges for a positive capacity set. By Theorem 5.29, it converges then for all values of z, in particular, h_z is holomorphic in some small polydisk $\{|x| < \rho',\ |z| < \rho''\}$.

Differentiating (5.33) in z, we see that the derivative $g(x) = \frac{\partial h_z(x)}{\partial z}\big|_{z=0}$ is a converging solution of the equation $(\frac{\partial g}{\partial x})Ax - Ag = f$, contrary to the assumption of the theorem. $\square$

Remark 5.33. The divergence assumption appearing in Theorem 5.32 can be easily achieved. Assume that A is a diagonal matrix with the spectrum $\{\lambda_j\}_1^n$ such that the differences $|\lambda_j - \langle\lambda,\alpha\rangle\,|$ decrease faster than *any* geometric progression $\rho^{|\alpha|}$ for any nonzero ρ. Assume also that the Taylor coefficients of f are bounded *from below* by *some* geometric progression. Then the series $\operatorname{ad}_A^{-1} f$ diverges.

It remains to observe that a set of positive measure is necessarily of positive capacity (Proposition 5.35), hence divergence guaranteed in the assumptions of Theorem 5.32, occurs for almost all z in the measure-theoretic sense, as stated in [**Ily79a**].

5H. Capacity and Bernstein inequality. The brief exposition below is based on [**PM01**] and the encyclopedic treatise [**Tsu59**].

Recall that the function $\ln|z-a|^{-1} = -\ln|z-a|$ is the electrostatic potential on the z-plane $\mathbb{C} \cong \mathbb{R}^2$, created by a unit charge at the point $a \in \mathbb{C}$ and harmonic outside a. If μ is a nonnegative measure (charge distribution) on the compact $K \Subset \mathbb{C}$, then its potential is the function represented by the integral $u_\mu(z) = \int_K \ln|z-a|^{-1}\,d\mu(a)$ and the energy of this measure is

$$E_\mu(K) = \iint_{K\times K} \ln|z-w|^{-1}\,d\mu(z)\,d\mu(w).$$

This energy can be either infinite for all measures, or $E_\mu(K) < +\infty$ for some nonnegative measures. In the latter case one can show that among all nonnegative measures normalized by the condition $\mu(K) = 1$, the (finite) minimal energy $E^*(K) = \inf_{\mu(K)=1} E_\mu(K)$ is achieved by a unique *equilibrium distribution* μ_K. The corresponding potential $u_K(z)$ is called the *conductor potential* of K.

Definition 5.34. The (harmonic, electrostatic) *capacity* of the compact K is either zero (when $E_\mu = +\infty$ for any charge distribution on K) or $\exp(-E^*(K)) > 0$ otherwise;

$$\varkappa(K) = \begin{cases} 0, & \text{if } \forall\mu\ E_\mu(K) = +\infty, \\ \sup\limits_{\mu(K)=1,\,\mu\geqslant 0} \exp(-E_\mu(K)), & \text{otherwise.} \end{cases} \tag{5.34}$$

Proposition 5.35. *Capacity of compact sets possesses the following properties*:

(1) *Countable union of zero capacity sets also has capacity zero.*

(2) $\varkappa(K) \geqslant \sqrt{\operatorname{mes}(K)/\pi e}$, *where* $\operatorname{mes}(K)$ *is the Lebesgue measure of* K, *in particular, if* K *is a set of positive measure, then* $\varkappa(K) > 0$.

(3) *If* K *is a Jordan curve of positive length, then* $\varkappa(K) > 0$.

Proof. All these assertions appear in [**Tsu59**] as Theorems III.8, III.10 and III.11 respectively. □

Proposition 5.36. *For compact sets of positive capacity, the conductor potential is harmonic outside* K, *and*

$$\begin{aligned} u_K \leqslant \varkappa^{-1}(K), \qquad u_K|_K = \varkappa^{-1}(K) \quad \textit{a.e.}, \\ u_K(z) = -\ln|z| + O(|z|^{-1}) \quad \textit{as } z \to \infty. \end{aligned} \tag{5.35}$$

Proof. [**Tsu59**, Theorem III.12] □

As a corollary, we conclude that for sets of the positive capacity there exists the Green function

$$G_K(z) = \varkappa^{-1}(K) - u_K(z) = \ln|z| + \varkappa^{-1}(K) + o(1) \quad \text{as } z \to \infty, \tag{5.36}$$

nonnegative on $\mathbb{C} \setminus K$, vanishing on K and asymptotic to the fundamental solution of the Laplace equation with the source at infinity.

Proof of Theorem 5.30 (Bernstein inequality). Since the assertion is invariant by multiplication by scalars, it is sufficient to prove for monic polynomials only.

Let $p(z) = z^r + \cdots$ be a monic polynomial of degree r. Consider the function

$$g(z) = \ln |p(z)| - \ln \|p\|_K - rG_K(z), \qquad z \in \mathbb{C} \smallsetminus K.$$

We claim that this function is nonpositive, $g \leqslant 0$ outside K. Indeed, g is negative near infinity since $g(z) = -\ln \|p\|_K - r\varkappa^{-1}(K) + o(1)$ as $z \to \infty$ by (5.36). On K we have the obvious inequality $\ln |p(z)| \leqslant \ln \|p\|_K$, and the Green function G_k has zero limit on K by (5.35). By construction, the function g is harmonic in $\mathbb{C} \smallsetminus K$ outside the isolated zeros of p where it tends to $-\infty$. By the maximum principle, the function g is nonpositive everywhere, $\ln |p(z)| \leqslant \ln \|p\|_K + rG_K(z)$ for all $z \in \mathbb{C} \smallsetminus K$. After passing to exponents this nonpositivity proves the theorem. □

Example 5.37. Assume that $K = [-1, 1]$ is the unit segment. Its complement is conformally mapped into the exterior of the unit disk $D = \{|w| < 1\}$ by the function $z = \frac{1}{2}(w + w^{-1})$, $w = z + \sqrt{z^2 - 1}$. The Green function G_D of the exterior is $\ln |w|$. Thus we obtain the explicit expression for G_K,

$$G_K = \ln \left| z + \sqrt{z^2 - 1} \right|,$$

which implies the classical form of the Bernstein inequality,

$$|p(z)| \leqslant \left| z + \sqrt{z^2 - 1} \right|^{\deg p} \max_{-1 \leqslant z \leqslant +1} |p(z)|. \tag{5.37}$$

Exercises and Problems for §5.

Problem 5.1. Prove that if h is a solution for the homogeneous homological equation (5.27) with a hyperbolic map f, then $H = h(x)\frac{\partial}{\partial x} \in \mathcal{D}(\mathbb{C}, 0)$ is a vector field that only by a constant factor differs from the generator of the self-map f: $f = \exp cH$, for some $c \in \mathbb{C}$.

Problem 5.2. Supply a detailed proof of the Poincaré theorem for self-maps (Theorem 5.17).

Exercise 5.3. Let $l \in \mathbb{R}$ be an irrational number whose rational approximations have only sub-exponential accuracy,

$$|l - \tfrac{p}{q}| > Ce^{-q^{1-\varepsilon}} \qquad \text{for some } C, \varepsilon > 0, \tag{5.38}$$

and $\mu = \exp 2\pi i l$. Prove that for any holomorphic right hand side f the homological equation

$$h \circ \mu - \mu h = f, \qquad f \in \mathcal{O}(\mathbb{C}, 0), \tag{5.39}$$

has an analytic (convergent) solution $h \in \mathcal{O}(\mathbb{C}, 0)$.

Exercise 5.4. Let $l \in \mathbb{R}$ be an irrational number which admits infinitely many exponentially accurate rational approximations p/q such that $|l - \frac{p}{q}| < e^{-q}$. Prove that for some right hand sides f the homological equation (5.39) has only divergent solutions (cf. with Remark 5.33).

Problem 5.5. Let $\mathbf{F} = F(x)x\frac{\partial}{\partial x} \in \mathcal{D}(\mathbb{C}^1, 0)$ be the germ of a holomorphic vector field at a singular point of multiplicity $k+1 \geqslant 2$ at the origin, $F(x) = x^{k+1}(1+o(1))$, and $\mathbf{F}' = \mathbf{F} + \mathbf{o}(x^{2k+1}) \in \mathcal{D}(\mathbb{C}^1, 0)$ is another such germ with the same $2k+1$-jet.

(i) Prove that these two germs are analytically equivalent if and only if two meromorphic 1-forms ω and ω', dual to $\mathbf{F}$ and $\mathbf{F}'$ respectively, are holomorphically equivalent (cf. with Remark 5.27).

(ii) Show that in the assumptions of the problem, the orders of the poles and the Laurent parts of the 1-forms ω and ω' coincide so that the difference $\omega - \omega'$ is holomorphic.

(iii) Passing to the primitives and denoting by $a_k, \dots, a_1, a_0$ the common Laurent coefficients of the forms ω, ω', prove that the equation

$$\frac{a_k}{y^k} + \cdots + \frac{a_1}{y} + a_0 \ln y + O(y) = \frac{a_k}{x^k} + \cdots + \frac{a_1}{x} + a_0 \ln x + O(z)$$

with holomorphic terms $O(y)$ and $O(x)$, admits a holomorphic solution $y = y(x)$ tangent to identity (substitute $y = ux$ and apply the implicit function theorem to the function $u(x)$ with $u(0) = 1$).

Problem 5.6 (Yet another proof of Schröder–Kœnigs theorem; cf. with [**CG93**])**.** Let $f \in \operatorname{Diff}(\mathbb{C}, 0)$ be a contracting hyperbolic holomorphic self-map, $f(z) = \lambda z + \cdots$, $|\lambda| < 1$, and $g(z) = \lambda z$ its linearization (the normal form).

Prove that the sequence of iterations $h_n = g^{-\circ n} \circ f^{\circ n}$ is defined and converges in some small disk around the origin. The limit $h = \lim h_n$ conjugates f and g.

Problem 5.7. Prove Theorem 5.5 along the same lines (M. Villarini).

6. Finitely generated groups of conformal germs

Thus far we have studied classification and certain dynamic properties of *single* germs of vector fields and biholomorphisms. However, in §2**C** we introduced an important invariant of foliation, the holonomy group of a leaf $L \in \mathcal{F}$ with nontrivial fundamental group $\pi_1(L, a)$, $a \in L$. By construction, the holonomy is a representation of $\pi_1(L, a)$ by conformal germs $\operatorname{Diff}(\tau, a)$, where τ is a cross-section to L at a, and the holonomy group G is identified with the image of that representation. Usually if the fundamental group of a leaf of a holomorphic foliation is finitely generated, then so is the group G. We will consider only the case of holomorphic foliations on complex 2-dimensional surfaces, thus dealing only with finitely generated subgroups of the group $\operatorname{Diff}(\mathbb{C}, 0)$ of conformal germs.

In this section we study classification problems for *finitely generated groups* of conformal germs and their dynamic properties, focusing on the properties which will be later used in §11 and §28. In much more detail the theory is treated in the recent monograph [**Lor99**].