

derivation \mathbf{F} . Respectively, the flow (germs of self-maps) will be sometimes denoted by the exponent, $\Phi^t = \exp(tF)$, of the corresponding vector field F .

Exercises and Problems for §1.

Exercise 1.1. Let $a \in U$ be a nonsingular point of a holomorphic vector field $F \in \mathcal{D}(U)$. A trajectory of the vector field is the projection of the graph of the solution into the domain of the field along the time axis.

Prove that the trajectory passing through a is either the line $x = a$, or can be represented as the graph of a function $y = \varphi_a(x)$ having an *algebraic* ramification point of some finite order ν . Express ν in terms of orders of the components of the field F at a .

Exercise 1.2. Let $P: (\mathbb{C}^n, 0) \rightarrow (\mathbb{C}^{n-1}, 0)$ be a holomorphic epimorphism (i.e., map of rank $n - 1$) constant along trajectories of an analytic vector field $F \in \mathcal{D}(\mathbb{C}^n, 0)$. Construct explicitly the rectifying chart for F .

Exercise 1.3. Prove that the space \mathcal{M} of functions satisfying the inequality (1.7), is indeed complete.

Exercise 1.4. Two linear vector fields in \mathbb{C}^n are holomorphically equivalent in some domains containing the origin. Prove that these fields are *linear* equivalent, i.e., that there exists a linear map $H \in \text{GL}(n, \mathbb{C})$ conjugating them.

Exercise 1.5. Prove that if two germs of vector fields at a singular point are analytically equivalent, then the eigenvalues of these fields at the singular point coincide.

Exercise 1.6. Prove that the vector field $F(z) = z^2 \frac{\partial}{\partial z}$ is holomorphic on the Riemann sphere $\mathbb{P}^1 = \mathbb{C} \cup \{\infty\}$. Compute the flow of this field.

Problem 1.7. Give a complete analytic classification of the holomorphic flows on the Riemann sphere \mathbb{P}^1 (i.e., construct a list, finite or infinite, of flows such that every holomorphic flow is analytically equivalent to one of the flows from the list, while any two different flows in the list are *not* holomorphically equivalent).

Exercise 1.8. Prove that the constant holomorphic vector fields $\frac{\partial}{\partial z}$ on the two tori $\mathbb{T}_1 = \mathbb{C}/(\mathbb{Z} + i\mathbb{Z})$ and $\mathbb{T}_2 = \mathbb{C}/(\mathbb{Z} + 2i\mathbb{Z})$, are not holomorphically equivalent.

2. Holomorphic foliations and their singularities

By the Existence/Uniqueness Theorem 1.1, any open connected domain $U \subseteq \mathbb{C}^n$ with a holomorphic vector field F defined on it, can be represented as the disjoint union of connected phase curves passing through all points of U . The Rectification Theorem 1.18 provides a local model for the geometric object called *foliated space* of simply *foliation*. A systematic treatment of foliations can be found, for instance, in [Tam92, CC03].

2A. Principal definitions. Speaking informally, a foliation is a partition of the phase space into a continuum of connected sets called *leaves*, which locally look as the family of parallel affine subspaces.

Definition 2.1. The *standard holomorphic foliation* of dimension n (respectively, of codimension m) of a polydisk $B = \{(x, y) \in \mathbb{C}^n \times \mathbb{C}^m : |x| < 1, |y| < 1\}$ is the representation of B as the disjoint union of n -disks, called (standard) *plaques*,

$$B = \bigsqcup_{|y| < 1} L_y, \quad L_y = \{|x| < 1\} \times \{y\} \subseteq B. \quad (2.1)$$

Definition 2.2. A holomorphic foliation \mathcal{F} of a domain $U \subset \mathbb{C}^{n+m}$ (or, more generally, a complex analytic manifold U of dimension $n + m$) is the partition $U = \bigsqcup_{\alpha} L_{\alpha}$ of the latter into the disjoint union *connected* subsets L_{α} , called *leaves*, which locally is biholomorphically equivalent to the standard holomorphic foliation by plaques.

The latter phrase means that each point $a \in U$ admits an open neighborhood $B' \ni a$ and a biholomorphism $H: B' \rightarrow B$ of B' onto the standard polydisk B , which sends the *local leaves*, the connected components of the intersections $L_{\alpha} \cap B'$, to the plaques of the standard foliation,

$$\forall \alpha \exists Y = Y(\alpha) : \quad H(L_{\alpha} \cap B') = \bigsqcup_{y \in Y(\alpha)} L_y. \quad (2.2)$$

Sometimes the local leaves will also be referred to as the *plaques* of the foliation near a point a : the plaques constitute biholomorphic images of n -disks, parameterized by a small m -disk. Note that different plaques may belong to the same leaf of the global foliation.

Remark 2.3. The definition of foliation admits several flavors. In the weakest settings the standard foliations are families of parallel balls slicing the real cylinder in \mathbb{R}^{n+m} (the formulas remain the same as in (2.1)), while the local equivalencies H are simply homeomorphisms or smooth maps of low or high differentiability (up to C^{∞} or even real analytic). In particular, we will call the *topological foliation* a partition of the space U into disjoint subsets L_{α} which is locally *homeomorphic* to the standard foliation (in the sense (2.2) with H being a homeomorphism).

Moreover, one can require *different* regularity of H along the leaves and in the transversal direction. We will not deal with such exotic cases until §28.

Remark 2.4 (important). The space of plaques of a foliation is naturally parameterized by points of a polydisk. Yet the index set $Y(\alpha)$ in (2.2) can be rather complicated (e.g., dense), since the global behavior of leaves outside

the ball B' can be rather complicated. Yet in all of our applications all sets $Y(\alpha)$ will be at most countable.

The global space of leaves may have a very complicated structure even topologically (non-Hausdorff), therefore for indexing the leaves we use “abstract” sets without any additional structure.

Definition 2.5. Two holomorphic foliations \mathcal{F} and \mathcal{F}' defined on the respective holomorphic manifolds U, U' , are called *holomorphically equivalent* or *topologically equivalent*, if there exists a biholomorphism $H: U \rightarrow U'$ (respectively, a homeomorphism) which maps (necessarily biholomorphically or homeomorphically, depending on the context) the leaves of \mathcal{F} to those of \mathcal{F}' : $H(L_\alpha) = L'_{\alpha'}$ for some indices α, α' .

Note that this definition is *global*.

Everywhere below U stands for a holomorphic manifold or an open domain in \mathbb{C}^n . The following result is an obvious reformulation of the Rectification theorem in the language of foliations.

Proposition 2.6. *For any holomorphic vector field $F \in \mathcal{D}(U)$ without singularities in U , the partition of U into maximal integral curves of F forms a holomorphic foliation \mathcal{F}_F of (complex) dimension 1 and codimension $n - 1$. \square*

We say that the foliation \mathcal{F}_F is generated by the vector field F . Speaking about foliations rather than about vector fields means that the parametrization of solutions by the (complex) time is to be ignored.

Proposition 2.7. *Two holomorphically equivalent vector fields $F \in \mathcal{D}(U)$ and $F' \in \mathcal{D}(U')$ generate two holomorphically equivalent one-dimensional foliations.*

Conversely, if the foliations $\mathcal{F}, \mathcal{F}'$ generated by two nonsingular vector fields, are holomorphically equivalent by a biholomorphism $H: U \rightarrow U'$, then there exists a nonvanishing holomorphic function $\rho \in \mathcal{O}(U)$ such that

$$\rho(x) \cdot H_*(x) \cdot F(x) = F'(H(x)), \quad \rho(x) \neq 0 \quad \text{in } U; \quad (2.3)$$

cf. with (1.26) and Definition 1.15.

Proof. The first assertion is obvious immediately. To prove the second, it is sufficient to show that two vector fields generating *the same* holomorphic one-dimensional foliation, differ by a nonvanishing holomorphic scalar factor ρ . This is obvious for the standard foliation: the first component must be nonzero while all other components are identically zero. \square

2B. Foliations and integrable distributions. For a given holomorphic foliation \mathcal{F} of dimension n and codimension m , the tangent spaces to leaves at different points are n -dimensional complex spaces in an obvious sense analytically depending on the point.

Such a geometric object is called *distribution*. To define formally subspaces analytically depending on parameters, one can choose between the language of holomorphic vector fields and that of holomorphic differential forms.

Definition 2.8. A (holomorphic nonsingular) n -dimensional distribution in a domain $U \subseteq \mathbb{C}^{n+m}$ is either

- a tuple of n holomorphic vector fields $F_1, \dots, F_n \in \mathcal{D}(U)$, linearly independent at every point of U , or
- tuple of m holomorphic 1-forms $\omega_1, \dots, \omega_m \in \Lambda^1(U)$, linearly independent at every point of U so that $\omega_1 \wedge \dots \wedge \omega_m \in \Lambda^m(U)$ is nonvanishing.

Two tuples of the same type $\{F_j\}$ and $\{F'_j\}$ (resp., $\{\omega_i\}$ and $\{\omega'_i\}$) define the same distribution, if $F'_j = \sum_k c_{jk}(x)F_k$, resp., $\omega'_i = \sum_k c'_{ik}(x)\omega_k$ for some holomorphic functions $c_{jk}(x), c'_{ik}(x)$. The forms and the fields defining the same distribution must be dual to each other, $\omega_i \cdot F_j = 0$ for all i, j .

A one-dimensional distribution is usually called a *line field*.

Clearly, any holomorphic foliation defines the corresponding tangent distribution of the same dimension. The converse in general is not true unless $n = 1$.

A holomorphic n -dimensional distribution is called *integrable* in U , if it is tangent to leaves of a nonsingular holomorphic foliation in U .

Theorem 2.9 (Frobenius integrability criteria). *A distribution defined by a tuple of holomorphic vector fields is integrable, if and only if the commutator of any two vector fields belongs to the same distribution, i.e., if*

$$[F_i, F_j] = \sum_{k=1}^n c_{ijk} F_k, \quad c_{ijk} \in \mathcal{O}(U). \quad (2.4)$$

A distribution defined by a tuple of holomorphic 1-forms is integrable, if and only if the ideal spanned by these forms in the exterior algebra $\Lambda^\bullet(U)$ over $\mathcal{O}(U)$, is closed by the exterior derivative, i.e., if

$$d\omega_i = \sum_{k=1}^m c'_{ik} \omega_k \wedge \eta_k, \quad \eta_k \in \Lambda^1(U), \quad c_{ik} \in \mathcal{O}(U). \quad (2.5)$$

We will not prove this theorem. Its proof can be derived from the local existence theorem for holomorphic vector fields in the same way as it is done, *mutatis mutandis*, in the C^∞ -smooth case in [War83].

Remark 2.10. The Frobenius integrability condition trivially holds for $n = 1$. On the other hand, from the real point of view the holomorphic vector field F corresponds to a 2-dimensional distribution generated by *two* vector fields $F_1 = F$ and $F_2 = iF$, $i = \sqrt{-1}$, in $\mathbb{R}^{2n} \cong \mathbb{C}^n$. The Frobenius integrability condition for this distribution reduces, as one can easily verify, to the Cauchy–Riemann identities between the real and imaginary parts of the components of the holomorphic vector field F .

Remark 2.11. In the (complex) 2-dimensional case where $U \subseteq \mathbb{C}^2$ that will be our principal object of studies later, the only nontrivial possibility is a one-dimensional distribution that is automatically integrable. It can be defined either by one vector field $F \in \mathcal{D}(U)$ or by one Pfaffian form $\omega \in \Lambda^1(U)$. For many reasons the Pfaffian presentation is more convenient.

2C. Holonomy. The notion of holonomy intends to be a replacement of the flow of the vector fields in the case where the natural parametrization of the solutions is absent or ignored.

Definition 2.12. A (parameterized) *cross-section* to a leaf L of a foliation \mathcal{F} of codimension m on U at a point $a \in U$ is a holomorphic map $\tau: (\mathbb{C}^m, 0) \rightarrow (U, a)$ transversal to L . Very often we identify the cross-section with the image of the map τ .

If \mathcal{F} is a *standard* foliation and τ, τ' any two cross-sections (at different, in general) points a, a' of the leaf, say $L_0 = \{y = 0\}$, then any other leaf L_α sufficiently close to L_0 intersects each cross-section exactly once. This defines in a unique way the holomorphic *correspondence map* $\Delta_{\tau, \tau'}: (\tau, a) \rightarrow (\tau', a')$: points with the same y -components are mapped into each other. In the charts on τ, τ' defined by the parameterizations, the correspondence map becomes the germ of a holomorphic map from $\text{Diff}(\mathbb{C}^m, 0)$.

The correspondence maps obviously satisfy the identity

$$\Delta_{\tau, \tau''} = \Delta_{\tau', \tau''} \circ \Delta_{\tau, \tau'} \tag{2.6}$$

for any three cross-sections τ, τ', τ'' to the same leaf of the standard foliation.

Taking a biholomorphic image of this construction, we arrive at the following conclusion. For any two cross-sections τ, τ' to two sufficiently close points on the same leaf, there exists a uniquely defined correspondence map $\Delta_{\tau, \tau'}$ between the cross-sections that satisfies the identity (2.6) for any third cross-section which is also sufficiently close.

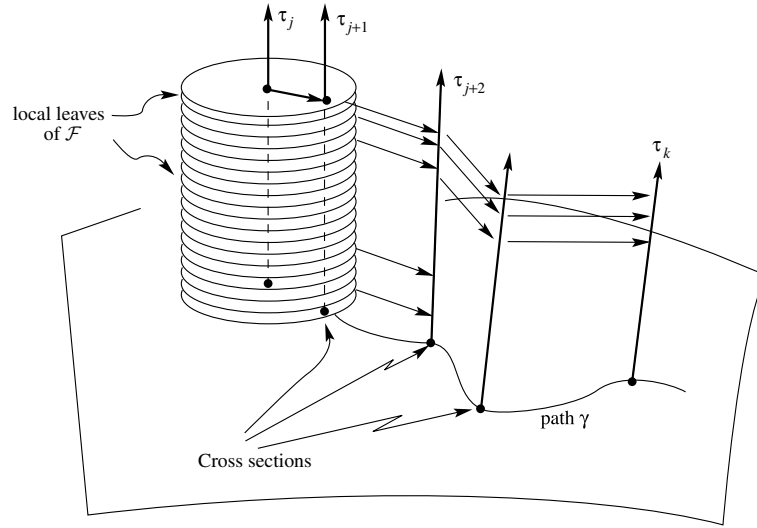


Figure I.2. Construction of the holonomy map for a foliation over a given path γ connecting two points on the leaf. The cross-sections τ_j are chosen close enough

Globalization of this construction associates the correspondence map not with just a pair of cross-sections to the same leaf, but rather with a *path* connecting the base points of these cross-sections. Let L be a leaf of a holomorphic foliation \mathcal{F} , τ, τ' two cross-sections cutting L at the points $a, a' \in L$, and $\gamma: [0, 1] \rightarrow L$ an (oriented) path connecting $a = \gamma(0)$ with $a' = \gamma(1)$.

Since the segment $[0, 1]$ and its image are compact, one can cover them by finitely many open sets U_j in such a way that in each set the foliation is locally trivial (biholomorphically equivalent to the standard foliation). One can insert between the cross-sections τ, τ' sufficiently many intermediate cross-sections τ_j , $j = 1, \dots, k$, $\tau_0 = \tau$, $\tau_k = \tau'$, at some intermediate points of the curve γ such that every two consecutive cross-sections τ_j, τ_{j+1} belong to the same domain U_j (for this purpose one has to choose $\tau_j \subset U_{j-1} \cap U_j$). Let $\Delta_{\tau_j, \tau_{j+1}}$ be the corresponding local correspondence maps as defined earlier. The composition

$$\Delta_\gamma = \Delta_{\tau_{k-1}, \tau_k} \circ \dots \circ \Delta_{\tau_0, \tau_1} : (\tau, a) \rightarrow (\tau', a') \quad (2.7)$$

is a holomorphic map (more precisely, a germ) from $\text{Diff}(\mathbb{C}^m, 0)$, also called the *correspondence map along the path* γ .

The identity (2.6) means that the correspondence map Δ_γ in fact does not depend on the choice of the intermediate cross-sections τ_j . Moreover, Δ_γ depends on the *homotopy class* of the path γ (with fixed endpoints) rather than on the path itself. Indeed, for another sufficiently close path

γ' connecting *the same endpoints*, we can choose cross-sections $\tau'_1, \dots, \tau'_{k-1}$ sufficiently close to the respective cross-sections τ_j for all $j = 1, \dots, k-1$ (the two extreme cross-sections coincide). Then one can use the identities (2.6) to show that the composition $\Delta_{\gamma'} = \Delta_{\tau'_{k-1}, \tau'_k} \circ \dots \circ \Delta'_{\tau'_0, \tau_1} : (\tau, a) \rightarrow (\tau', a')$ coincides with Δ_γ , since $\tau'_0 = \tau_0$ and $\tau'_k = \tau_k$.

Remark 2.13. The construction of holonomy maps corresponds to what in the classical parlance was called “continuation of solutions of differential equations over a path”: a specific solution (corresponding to the leaf) was explicitly or implicitly singled out together with a certain path on it, and all nearby solutions were “continued over the path” on the selected solution.

Choosing another pair of cross-sections at the same endpoints (or another parametrization of the same cross-sections) results in composition of Δ_γ with two biholomorphisms from left and right, so using suitable charts, one can always bring any particular correspondence map Δ_γ to be the identity map. The situation changes completely if there is more than one homotopically distinct path connecting the same endpoints, or, what is the same, when one considers *closed paths*.

Let $a \in L$ be a point on the leaf L of a holomorphic foliation, $\tau: (\mathbb{C}^m, 0) \rightarrow (U, a)$ a cross-section at a , and $\gamma \in \pi_1(L, a)$ a closed loop considered modulo the homotopic equivalence.

Definition 2.14. The *holonomy self-map* $\Delta_\gamma: (\tau, a) \rightarrow (\tau, a)$ is the holomorphic holonomy correspondence map associated with a closed path $\gamma \in \pi_1(L, a)$.

The *holonomy group* of the foliation \mathcal{F} along the leaf $L \in \mathcal{F}$ is the image of the fundamental group $\pi_1(L, a)$ in the group of germs of holomorphic self-maps $\text{Diff}(\tau, a)$.

The holonomy group is defined as a subgroup in $\text{Diff}(\mathbb{C}^m, 0)$ modulo a simultaneous conjugacy of all holonomy maps, independently of the choice of the cross-section τ or even the base point $a \in L$. It is an obvious invariant of a foliation which carries almost all information on behavior of leaves of the foliation, adjacent to L .

Proposition 2.15. *Assume that two holomorphic foliations $\mathcal{F}, \mathcal{F}'$ are topologically or holomorphically conjugate by a homeomorphism (resp., biholomorphism) H . If $L \in \mathcal{L}$ is a leaf mapped by H into a leaf $L' \in \mathcal{F}'$, then for any choice of the points $a \in L, a' \in L'$ and the corresponding cross-sections τ, τ' the corresponding holonomy groups $G \subset \text{Diff}(\tau, a)$ and $G' \subset \text{Diff}(\tau', a')$ are topologically (resp., holomorphically) conjugate: there exists the germ of a map $h: (\tau, a) \mapsto (\tau', a')$, holomorphic or continuous respectively, such*

that h conjugates each element of g with some element $g' \in G'$ and respects the group law.

Proof. Let τ be a cross-section to L at a and $\tau' = H(\tau)$ (with the induced chart), then the assertion is a tautology: the restriction $h = H|_{\tau}$ realizes the required conjugacy between G and G' . Any other choice of a' and τ' results in replacing G' by a holomorphically conjugate group. \square

However, the inverse statement is in general wrong (see Exercise 2.10).

Definition 2.16. Let \mathcal{F} be a holomorphic foliation on a complex manifold U , and $B \subseteq U$ an arbitrary subset. The *saturation* of B by leaves of \mathcal{F} is the union of all leaves that intersect B :

$$\text{Sat}(B, \mathcal{F}) = \bigcup_{L \in \mathcal{F}, L \cap B \neq \emptyset} L.$$

In general, saturations of even simple sets can be rather complicated. Yet some basic things can be guaranteed. The following can be considered as a generalization of the theorem on continuous dependance of solutions of differential equations on initial conditions.

Lemma 2.17. *Saturation of an open set is open. In particular, saturation of a neighborhood of any point on each leaf contains an open neighborhood of the leaf.* \square

From this observation we can derive a corollary that will be used later. Let $G \subset \text{Diff}(\tau, a)$ be a finitely generated subgroup. A germ of an analytic function $u \in \mathcal{O}(\tau, a)$ is called G -invariant, if $u \circ g = u$ for all germs of self-maps $g \in G$.

Lemma 2.18. *Any germ of a holomorphic function $u \in \mathcal{O}(\tau, a)$ which is invariant by the holonomy group $G \subseteq \text{Diff}(\tau, a)$, uniquely extends as a holomorphic function defined in some open neighborhood V of the leaf L and constant along all leaves of the foliation \mathcal{F} in V .*

Proof. Let $a' \in L$ be any point on L , connected by a path $\gamma: [0, 1] \rightarrow L$ with the base point a . The holonomy map $\Delta_{a, a'}$ allows us to translate (analytically continue) the germ u , considered as a function from $\mathcal{O}(U, a)$ constant along the local plaques of \mathcal{F} , to the germ $u' \in \mathcal{O}(U, a')$, also constant along the local plaques. This extension depends on the choice of the path γ , yet for a different choice of this path γ' the result will differ by the continuation of the germ $u \circ g$, where g is the holonomy map associated with the loop $\gamma' \circ \gamma^{-1} \in \pi_1(L, a)$. Yet since u by assumption is G -invariant, the result will be the same and thus correctly defined for an arbitrary point $a' \in L$. \square

Remark 2.19. Most holonomy groups do not admit nonconstant invariant functions. Exceptions correspond to *integrable foliations*; see §11.

2D. Singular foliations. The holonomy group may be nontrivial only for a leaf of the foliation which has a nontrivial fundamental group. Such leaves, in general difficult to find for arbitrary holomorphic foliations, can be easily found for *foliations with singularities*, or *singular foliations*. Starting from this moment, we consider only one-dimensional foliations unless explicitly stated otherwise.

A holomorphic vector field $F \in \mathcal{D}(U)$ defines a nonsingular holomorphic foliation on the complement to its singular locus $\Sigma = \Sigma_F = \{x \in U : F(x) = 0\}$ by Proposition 2.6. This singular locus can be an arbitrary analytic subset of U . However, very often the foliation can be extended from U on a bigger open subset eventually containing a part of Σ .

If $U \subset U'$ are two domains and \mathcal{F}' a foliation on the larger domain, then \mathcal{F}' can be restricted on U : by definition, this means the foliation whose leaves are *connected components* of the intersections $L'_\alpha \cap U$ for all leaves $L'_\alpha \in \mathcal{F}'$.

Theorem 2.20. *Let U be a connected open domain in \mathbb{C}^n and $0 \neq F \in \mathcal{D}(U)$ a holomorphic vector field with the singular locus $\Sigma \subset U$.*

Then there exists an analytic subset $\Sigma' \subseteq \Sigma$ of complex codimension ≥ 2 in U and the foliation \mathcal{F}' of $U \setminus \Sigma'$ whose restriction on $U \setminus \Sigma$ coincides with the foliation generated by the initial vector field F .

Proof. The assertion needs the proof only when Σ is an analytic hypersurface (a complex analytic set of codimension 1).

Consider an arbitrary *smooth point* $a \in \Sigma$ of the singular locus Σ : nonsmooth points already form an analytic subset $\Sigma_1 \subset \Sigma$ of codimension ≥ 2 in U . Locally near this point Σ can be described by one equation $\{f = 0\}$ with f holomorphic and $df(a) \neq 0$. Let $\nu \geq 1$ be the maximal power such that all components F_1, \dots, F_n of the vector field F are divisible by f^ν . By construction, the vector field $f^{-\nu} F$ extends analytically on Σ near a and its singular locus is a *proper* analytic subset $\Sigma_2 \subset \Sigma$ (locally near a). Since the germ of Σ at a is smooth hence irreducible, such a subset necessarily has codimension ≥ 2 relative to the ambient space.

The union $\Sigma' = \Sigma_1 \cup \Sigma_2$ has codimension ≥ 2 and in $U \setminus \Sigma'$ the field locally represented as $f^{-\nu} F$ is nonsingular and thus defines a holomorphic foliation \mathcal{F}' extending \mathcal{F} on the neighborhood of all points of Σ . \square

Remark 2.21. If U is two-dimensional, the holomorphic vector field F can be replaced by the distribution defined by an appropriate holomorphic 1-form $\omega \in \Lambda^1(U)$ with the singular locus Σ which consists of isolated points

only (the singular locus of a holomorphic 1-form is the common zero of its coefficients).

Theorem 2.20 means that when speaking about holomorphic foliations with singularities, generated by holomorphic vector fields, one can always assume that the singular locus has codimension ≥ 2 ; in particular, singularities of holomorphic foliations on the plane (and more generally, on holomorphic surfaces) are *isolated points*. The inverse statement is also true, as was observed in [Ily72b].

Theorem 2.22 ([Ily72b]). *Assume that $\Sigma \subset U \subseteq \mathbb{C}^n$ is an analytic subset of codimension ≥ 2 and \mathcal{F} a holomorphic nonsingular 1-dimensional foliation of $U \setminus \Sigma$ which does not extend on any part of Σ .*

Then near each point $a \in \Sigma$ the foliation \mathcal{F} is generated by a holomorphic vector field F with the singular locus Σ .

Proof. One can always assume that the local coordinates near a are chosen so that the line field tangent to leaves of \mathcal{F} , is not everywhere parallel to the coordinate x_1 -plane. Then this line field is spanned by the *meromorphic* vector field $G = (1, G_2, \dots, G_n)$, where $G_j \in \mathcal{M}(U \setminus \Sigma)$ are *meromorphic* functions in $U \setminus \Sigma$. By E. Levi's theorem, any meromorphic function can be meromorphically extended on analytic subsets of codimension 1 [GH78, Chapter III, §2]. Therefore we may assume that G_j are in fact meromorphic in U . Decreasing if necessary the size of U , each G_j can be represented as the ratio $G_j = P_j/Q_j$, where $P_j, Q_j \in \mathcal{O}(U)$ are holomorphic in U and the representation is irreducible.

Let $\Sigma_j = \{P_j = Q_j = 0\}$, $j = 2, \dots, n$: by irreducibility, Σ_j is of codimension ≥ 2 , so $\bigcup_{j \geq 2} \Sigma_j$ is also of codimension ≥ 2 . Multiplying the field G by the product of denominators $Q_2 \cdots Q_n$, we obtain a holomorphic vector field tangent to the same foliation; cancelling a nontrivial common factor for the components of this field as in Theorem 2.20, we arrive at yet another holomorphic field F , also tangent to \mathcal{F} , such that the singular locus $\Sigma' = \text{Sing}(F)$ of this field has codimension ≥ 2 .

It remains to show that the singular locus Σ' *coincides* with Σ locally in U . In one direction it is obvious: if Σ' is *smaller* than Σ , this means that \mathcal{F} is analytically extended as a nonsingular holomorphic foliation to some parts of Σ , contrary to the assumption that Σ is the minimal singular locus. Assume that Σ' is *larger* than Σ , i.e., there exists a nonsingular point $b \in U \setminus \Sigma$ of \mathcal{F} , at which F vanishes. Since the foliation \mathcal{F} is biholomorphically equivalent to the standard foliation near b , in the suitable chart F is parallel to the first coordinate axis, so that singular points of F are zeros of its first component. On the other hand, by construction Σ' is of codimension ≥ 2 and hence

cannot be the zero locus of any holomorphic function. The contradiction proves that $\Sigma' \cap U$ cannot be larger than $\Sigma \cap U$. \square

Example 2.23. The vector field $\frac{\partial}{\partial x} + e^{1/x} \frac{\partial}{\partial y}$ is analytic outside the line $\Sigma = \{x = 0\}$ of codimension 1 on the plane and defines a holomorphic foliation in $\mathbb{C}^2 \setminus \Sigma$. This foliation cannot be defined by a vector field holomorphically extendable on Σ , which shows that the condition on the codimension in Theorem 2.22 cannot be relaxed.

Together Theorems 2.20 and 2.22 motivate the following concise definition. Since we will never consider in this book holomorphic foliations of dimension other than 1, this is explicitly included in the definition.

Definition 2.24. A *singular holomorphic foliation* in a domain U (or a complex analytic manifold) is a holomorphic foliation \mathcal{F} with complex one-dimensional leaves in the complement $U \setminus \Sigma$ to an analytic subset Σ of codimension ≥ 2 , called the *singular locus* of \mathcal{F} .

Usually we will assume that the singular locus Σ is maximal, i.e., the foliation cannot be analytically extended on any set larger than $U \setminus \Sigma$.

The second part of Proposition 2.7 motivates the following important definition.

Definition 2.25. Two holomorphic vector fields $F \in \mathcal{D}(U)$, $F' \in \mathcal{D}(U')$ with singular loci Σ, Σ' of codimension ≥ 2 are *holomorphically orbitally equivalent* if the singular foliations $\mathcal{F}, \mathcal{F}'$ they generate, are holomorphically equivalent, i.e., there exists a biholomorphism $H: U \rightarrow U'$ which maps Σ into Σ' and is a biholomorphism of foliations outside these loci.

Proposition 2.7 remains valid also for *singular* holomorphic foliations: if two such foliations are holomorphically equivalent, then the corresponding vector fields are orbitally equivalent, i.e., related by the identity (2.3) with the holomorphic function ρ nonvanishing in U .

Indeed, from Proposition 2.7 it follows that for the holomorphically orbitally equivalent fields there exists a holomorphic function ρ satisfying (2.3) and nonvanishing outside $\Sigma = \text{Sing}(F)$. Since Σ has codimension ≥ 2 , ρ must be nonvanishing *everywhere* on U .

Changing only one adjective in Definition 2.25 (requiring that H be merely a homeomorphism), we obtain the definition of *topologically* orbitally equivalent vector fields. This weaker equivalence cannot be translated into a formula similar to (2.3), since homeomorphisms in general do not act on the vector fields.

2E. Complex separatrices. Foliations with isolated singularities may have *multiply-connected* leaves, i.e., leaves with a nontrivial holonomy group.

Recall that a (singular) analytic curve $S \subset U$ is a complex analytic set of complex dimension 1 at its smooth points. Intrinsic structure of *irreducible* components of analytic curves is relatively easy. This result can be found, e.g., in [Chi89, §6].

Theorem 2.26. *The germ of an irreducible analytic curve $S \subset (\mathbb{C}^n, 0)$ admits a holomorphic injective map*

$$\gamma: (\mathbb{C}^1, 0) \rightarrow (\mathbb{C}^n, 0), \quad t \mapsto \gamma(t) \in S. \quad (2.8)$$

The map γ is called *local uniformization*, or *local parametrization* of analytic curves. It is obviously nonconstant, and without loss of generality one may assume that the derivative $\frac{d}{dt}\gamma(t)$ is nonvanishing outside the origin $t = 0$. The local parametrization is defined uniquely modulo a biholomorphism: for any other injective parametrization γ' there exists $h \in \text{Diff}(\mathbb{C}^1, 0)$ such that $\gamma' = \gamma \circ h$ (cf. with Exercise 2.1).

Let \mathcal{F} be a singular holomorphic foliation on an open domain U with the singular locus Σ .

Definition 2.27. A *complex separatrix* of a singular holomorphic foliation \mathcal{F} at a singular point $a \in \text{Sing}(\mathcal{F})$ is a local leaf $L \subset (U, a) \setminus \Sigma$ whose closure $L \cup \{a\}$ is the germ of an analytic curve.

Since the leaves are by definition connected, the closure is irreducible (as a germ) at any of its points, hence (by the above uniformization arguments) the complex separatrix is topologically a punctured disk near the singularity. The fundamental group of the separatrix is nontrivial (infinite cyclic), thus the holomorphic map generating the *local holonomy group* is an invariant of the singular foliation. Note that the leaves are naturally oriented by their complex structure, so the loop generating the local fundamental group is uniquely defined modulo free homotopy.

In other words, every singular point that admits a complex separatrix, produces at least one holomorphic germ of a self-map that is an analytic invariant of the foliation. Later, in §14 we will show that *every* planar foliation (on a complex 2-dimensional manifold) has at least one separatrix through each singularity. Besides, by blow-up (desingularization) and Poincaré compactification, two related operations discussed in detail in §8 and §25A respectively, one can often *create* multiply-connected leaves of singularities *extending* a given singular foliation.

The rest of this section consists of a few examples important for future applications.

Example 2.28. Consider first the singular foliation spanned by a *diagonal* linear system

$$\dot{x} = Ax, \quad A = \text{diag}\{\lambda_1, \dots, \lambda_n\}, \quad \lambda_j \neq 0. \quad (2.9)$$

This foliation has an isolated singularity (of codimension n) at the origin, and all coordinate axes are complex separatrices.

Consider the first coordinate axis $S_1 = \{x_2 = \dots = x_n = 0\}$ and the separatrix $L_1 = S_1 \setminus \{0\}$. The loop $\gamma = \{|x_1| = 1\}$ parameterized counterclockwise is the canonical generator of L_1 . Choose the affine hyperplane $\tau = \{x_1 = 1\} \subset \mathbb{C}^n$ as the cross-section to S_1 at the point $(1, 0, \dots, 0) \in S_1$. A solution of the system (the parameterized leaf of the foliation) passing through the point $(1, b_2, \dots, b_n) \in \tau$ is as follows:

$$\mathbb{C}^1 \ni t \mapsto x(t) = (\exp \lambda_1 t, b_2 \exp \lambda_2 t, \dots, b_n \exp \lambda_n t) \in \mathbb{C}^n.$$

The image of the straight line segment $[0, 2\pi i/\lambda_1] \subset \mathbb{C}$ on the t -plane coincides with the loop γ when $b = 0$ (i.e., on the separatrix S_1) and is uniformly close to this loop on all leaves near S_1 . The endpoints $x(2\pi i/\lambda_1)$ all belong to τ and hence the holonomy map $M_1: \mathbb{C}^{n-1} \rightarrow \mathbb{C}^{n-1}$ is linear diagonal,

$$b \mapsto M_1 b, \quad M_1 = \text{diag}\{2\pi i \lambda_j / \lambda_1\}_{j=2}^n. \quad (2.10)$$

The other holonomy maps M_k for the canonical loops on the separatrices S_k parallel to the k th axis, are obtained by obvious relabelling of the indices.

Particular cases of this result are of special importance.

Example 2.29. Consider an *integrable* planar foliation given by the Pfaffian equation $\omega = 0$ with an *exact* form $\omega = du$, $u \in \mathcal{O}(\mathbb{C}^2, 0)$. If u has a Morse critical point, then in suitable analytic coordinates (x, y) the germ u takes the form $u = xy$, hence the foliation is given by the *linear* form $x dy + y dx = 0$ corresponding to the vector field $\dot{y} = y$, $\dot{x} = -x$. The holonomy operators corresponding to the two coordinate axes, are both *identical*.

Integrable foliations with more degenerate singularities will be treated in detail in §11.

Example 2.30. Let $n = 2$. Consider the vector field $F = (x + y) \frac{\partial}{\partial x} + y \frac{\partial}{\partial y}$ corresponding to a linear vector field with a nontrivial Jordan matrix. The corresponding singular foliation has only one complex separatrix, the punctured axis $S = \{y = 0\}$.

Consider the standard cross-section $\tau = \{x = 1\}$. Solutions of the differential equation with the initial condition (x_0, y_0) can be written explicitly,

$$x(t) = (x_0 + t y_0) \exp t, \quad y = y_0 \exp t.$$

Let $t(y_0)$ be another moment of complex time when the solution close to the separatrix again crosses τ after continuing along a path close to the standard loop on the separatrix; by definition, this means that we consider the initial point with $x_0 = 1$ and $x(t(y_0)) \equiv 1$, i.e., $1 + t(y_0)y_0 = 1/\exp t(y_0)$.

If the holonomy map is linear, then $y(t(y_0)) = \lambda y_0$ identically in y_0 , i.e., $\exp t(y_0) = \lambda$ is a constant. Substituting this into the previous identity, we obtain $1 + t(y_0)y_0 = 1/\lambda$. This is impossible in the limit $y_0 \rightarrow 0$ unless $\lambda = 1$. On the other hand, $\lambda = 1$ is also impossible since $t(y_0) \not\equiv 0$.

Thus the holonomy map cannot be linear. The principal term of this map in a more general setting is computed in Proposition 27.14.

This example shows that a linear foliation may have nonlinear (and even nonlinearizable) holonomy.

2F. Suspension of a self-map. The construction of holonomy associates with any loop γ on a leaf $L \in \mathcal{F}$ of a holomorphic foliation \mathcal{F} the holomorphic self-map Δ_γ . Very often the inverse problem appears: given an invertible holomorphic self-map f , construct a foliation for which this self-map would be the holonomy, associated with a loop on a leaf.

We will show that in absence of additional constraints on the phase space M and the leaf L , this problem is always trivially solvable. The construction is well known in the real analysis as *suspension* of a map to a flow.

Theorem 2.31. *Any biholomorphic germ $f \in \text{Diff}(\mathbb{C}^n, 0)$ can be realized as the holonomy map along a loop on the leaf of a holomorphic foliation on an $(n + 1)$ -dimensional holomorphic manifold M^{n+1} .*

Construction of the foliation. For simplicity we discuss only the case $n = 1$: the general case requires only minimal modifications.

Consider the segment $[0, 1] \subset \mathbb{C}$ and let U be its ε -neighborhood, $\varepsilon < \frac{1}{2}$. In the Cartesian product $\widetilde{M} = U \times (\mathbb{C}, 0)$ with the coordinates (z, w) consider the trivial foliation \mathcal{F}_0 by “horizontal lines” $\{w = \text{const}\}$.

Any self-map from $f \in \text{Diff}(\mathbb{C}^1, 0)$ can be considered as a map $f: (\tau_0, 0) \rightarrow (\tau_1, 0)$, $w \mapsto f(w)$, between the cross-sections $\tau_0 = \{z = 0\}$ and $\tau_1 = \{z = 1\}$. The latter can be extended as a holomorphic invertible map $\mathbf{f}: (z, w) \mapsto (z + 1, f(w))$ between the open sets $M_0 = \{|z| < \varepsilon\} \times (\mathbb{C}, 0) \subset \widetilde{M}$ and $M_1 = \{|z - 1| < \varepsilon\} \times (\mathbb{C}, 0) \subset \widetilde{M}$. By construction, this map preserves the restriction of the foliation \mathcal{F}_0 on the open sets M_i .

The quotient space $M = \widetilde{M}/\mathbf{f}$ is defined as the topological space obtained from \widetilde{M} by identification of all points a and $\mathbf{f}(a)$. This space inherits the structure of an (abstract) holomorphic manifold (the charts are inherited from those on \widetilde{M}). Moreover, since \mathbf{f} preserves the foliation, the quotient manifold M carries a well defined foliation \mathcal{F} . Two different cross-sections $\tau_0, \tau_1 \subset \widetilde{M}$ after identification become a single cross-section τ to the leaf L of the foliation \mathcal{F} obtained from the zero leaf $\{w = 0\} \in \mathcal{F}_0$, and the segment $[0, 1]$ on this leaf becomes a closed loop on L . The holonomy of the foliation

\mathcal{F} , associated with the loop $\gamma \subset L$, by construction coincides with the map f which is transformed into the self-map. \square

The construction can be modified by a number of ways, while keeping the principal idea the same. If \widetilde{M} is a manifold with a foliation \mathcal{F}_0 on it, and $\mathbf{f}: M_0 \rightarrow M_1$ is a biholomorphic map between open subsets of \widetilde{M} , which is an automorphism of the foliation \mathcal{F}_0 , then the quotient space $M = \widetilde{M}/\mathbf{f}$ is a new manifold with a different topology, which carries a holomorphic foliation on it.

Exercises and Problems for §2.

Exercise 2.1. Let $S \subset (\mathbb{C}^n, 0)$ be the germ of an irreducible analytic curve and γ an injective analytic parametrization. Prove that any other holomorphic map $\gamma': (\mathbb{C}^1, 0) \rightarrow (\mathbb{C}^n, 0)$ with the range in S differs from γ by a holomorphic map $h: (\mathbb{C}^1, 0) \rightarrow (\mathbb{C}^1, 0)$ so that $\gamma' = \gamma \circ h$.

Problems 2.2–2.7 together constitute a proof of the Frobenius Theorem 2.9.

Problem 2.2. Prove that vector fields generating an integrable distribution, are *in involution*, i.e., always satisfying condition (2.4).

Prove that Pfaffian forms generating an integrable distribution, are in involution, i.e., satisfy the conditions (2.5).

Problem 2.3. Prove that two holomorphic vector fields $F, F' \in \mathcal{D}(M)$ on a holomorphic manifold M , have identically zero commutator, $[F, F'] \equiv 0$, if and only if their flows $\exp(tF), \exp(t'F') \in \text{Diff}(M)$ commute for all complex values of $t, t' \in \mathbb{C}$.

Formulate and prove an analog of this result for *incomplete* vector fields (i.e., when the flows are not globally defined for all values of t, t' , as in the case where $U \subseteq \mathbb{C}^2$ is a noninvariant planar domain).

Problem 2.4. Prove that any tuple of everywhere linearly independent commuting vector fields generates an integrable distribution tangent to leaves of a holomorphic foliation.

Problem 2.5. Let F_1, \dots, F_k be holomorphic everywhere linearly independent vector fields in involution (i.e., satisfying condition (2.4)).

Construct another tuple of holomorphic vector fields F'_1, \dots, F'_k spanning the same distribution, such that the fields $[F'_i, F'_j] \equiv 0$ for all $1 \leq i, j \leq k$.

Prove that vector fields in involution generate an integrable distribution.

Problem 2.6. Prove that for any differential 1-form ω and two vector fields F, G on a manifold M ,

$$d\omega(F, G) = F\omega(G) - G\omega(F) - \omega([F, G]) \quad (2.11)$$

(the right hand side contains the evaluation of ω on the fields F, G and $[F, G]$ and their derivatives along G and F).

Problem 2.7. Prove that a tuple of everywhere linearly independent 1-forms satisfying (2.5), defines an integrable distribution.

Exercise 2.8. Prove that a nonvanishing Pfaffian form ω in \mathbb{C}^3 defines an integrable distribution, if and only if $\omega \wedge d\omega = 0$.

Problem 2.9. Prove that each holonomy operator g corresponding to any separatrix of an integrable foliation $du = 0$ with an analytic potential $u \in \mathcal{O}(x, y)$, is periodic: some iterated power of g is identity.

Exercise 2.10. Construct two foliations having leaves with holomorphically conjugated holonomy groups, which are themselves not holomorphically conjugate in neighborhoods of the leaves.

Exercise 2.11. Is it always possible to rectify *simultaneously* two nonsingular vector fields? Two *commuting* nonsingular vector fields? Give a simple sufficient condition guaranteeing such simultaneous rectification.

Exercise 2.12. Consider the foliation $\{\omega = 0\}$ on $\mathbb{C}^2 = \{(z, t)\}$ defined by a meromorphic Pfaffian 1-form

$$\omega = \frac{dz}{z} - \sum_{j=0}^n \frac{\lambda_j dt}{t - a_j}, \quad \lambda_j \in \mathbb{C}, \quad \sum_0^n \lambda_j = 0,$$

and its extension on $\mathbb{C} \times \mathbb{P}^1$.

Prove that the projective line $L = \{0\} \times \mathbb{P}^1$ is a separatrix of this foliation carrying singular points $(0, a_j)$, $j = 0, \dots, n$. Compute the holonomy group of the leaf $L \setminus (\text{singular points})$.

Exercise 2.13. The same question about the foliation on $\mathbb{C}^m \times \mathbb{P}^1$ defined by the vector Pfaffian form

$$dz - \Omega z = 0, \quad \Omega = \sum_0^n \frac{A_j dt}{t - a_j},$$

where $A_j \in \text{Mat}(m, \mathbb{C})$ are *commuting* matrix residues of the meromorphic matrix 1-form Ω .

Problem 2.14. Consider the Riccati equation

$$\frac{dz}{dt} = a(t)z^2 + b(t)z + c(t), \quad a, b, c \in \mathcal{M}(\mathbb{P}) \cong \mathbb{C}(t), \quad (2.12)$$

with meromorphic coefficients a, b, c having poles only on the finite point set $\Sigma \subseteq \mathbb{P}$. Is it true that solutions of this equation can be continued along any path on the t -plane, avoiding the singular locus Σ ?

Prove that equation (2.12) defines a singular holomorphic foliation \mathcal{F} on the compactified phase space $\mathbb{P}^1 \times \mathbb{P}^1$, which is transversal to any “vertical” projective line $\{t = a\}$, $a \notin \Sigma$. Show that each leaf of \mathcal{F} can be continued over any path in the t -sphere, avoiding the singular locus. Prove that the induced transformation between any two cross-sections $\{t = a\} \times \mathbb{P}^1$ and $\{t = b\} \times \mathbb{P}^1$, $a, b \notin \Sigma$, is a well-defined Möbius transformation (fractional linear map $z \mapsto \frac{\alpha z + \beta}{\gamma z + \delta}$ with $\alpha\delta - \beta\gamma \neq 0$). Does \mathcal{F} always possess a separatrix?

Exercise 2.15. How many separatrices a *homogeneous* vector field of degree r on \mathbb{C}^2 may have? How many separatrices a *generic* homogeneous vector field has?