Chapter I

Normal forms and desingularization

1. Analytic differential equations in the complex domain

For an open domain $U \subseteq \mathbb{C}^n$ we denote by $\mathcal{O}(U)$ the complex linear space of functions holomorphic in U (see Appendix). The space of vector-valued holomorphic functions is denoted by

$$\mathcal{O}^m(U) = \underbrace{\mathcal{O}(U) \times \cdots \times \mathcal{O}(U)}_{m \text{ times}} = \mathcal{O}(U) \otimes_{\mathbb{C}} \mathbb{C}^m.$$

1A. Differential equations, solutions, initial value problems. Let $U \subseteq \mathbb{C} \times \mathbb{C}^n$ be an open domain and $F = (F_1, \dots, F_n) \colon U \to \mathbb{C}^n$ a holomorphic vector function. An *analytic ordinary differential equation* defined by F on U is the vector equation (or the *system* of n scalar equations)

$$\frac{dx}{dt} = F(t, x), \qquad (t, x) \in U \subseteq \mathbb{C} \times \mathbb{C}^n, \quad F \in \mathcal{O}^n(U). \tag{1.1}$$

The *solution* of this equation is a parameterized holomorphic curve, the holomorphic map $\varphi = (\varphi_1, \dots, \varphi_n) \colon V \to \mathbb{C}^n$, defined in an open subset $V \subseteq \mathbb{C}$, whose graph $\{(t, \varphi(t)) \colon t \in V\}$ belongs to U and whose complex "velocity vector" $\frac{d\varphi}{dt} = \left(\frac{d\varphi_1}{dt}, \dots, \frac{d\varphi_n}{dt}\right) \in \mathbb{C}^n$ at each point t coincides with the vector $F(t, \varphi(t)) \in \mathbb{C}^n$.

The graph of φ in U is called the *integral curve*. From the real point of view it is a 2-dimensional smooth surface in $\mathbb{R}^{2n+2}$. Note that from the beginning we consider only holomorphic solutions which may be, however, defined on domains of different size.

The equation is *autonomous*, if F is independent of t. In this case the image $\varphi(V) \subseteq \mathbb{C}^n$ is called the *phase curve*. Any differential equation (1.1) can be "made" autonomous by adding a fictitious variable $z \in \mathbb{C}$ governed by the equation $\frac{dz}{dt} = 1$.

If $(t_0, x_0) = (t_0, x_{0,1}, \dots, x_{0,n}) \in U$ is a specified point, then the *initial value problem*, sometimes also called the *Cauchy problem*, is to find an integral curve of the differential equation (1.1) passing through the point (t_0, x_0), i.e., a solution satisfying the condition

$$\varphi\colon V \to \mathbb{C}^n, \qquad \varphi(t_0) = x_0 \in \mathbb{C}^n. \tag{1.2}$$

In what follows we will often denote by a dot the derivative with respect to the complex variable t, $\dot{x}(t) = \frac{dx}{dt}(t)$.

The first fundamental result is the local existence and uniqueness theorem.

Theorem 1.1. *For any holomorphic differential equation* (1.1) *and every point* $(t_0, x_0) \in U$ *there exists a sufficiently small polydisk* $D_\varepsilon = \{|t - t_0| < \varepsilon,\ |x_j - x_{0,j}| < \varepsilon,\ j = 1, \dots, n\} \subseteq U$, *such that the solution of the initial value problem* (1.2) *exists and is unique in this polydisk.*

This solution depends holomorphically on the initial value $x_0 \in \mathbb{C}^n$ *and on any additional parameters, provided that the vector function* F *depends holomorphically on these parameters.*

From the real point of view, Theorem 1.1 asserts existence of $2n$ functions of *two* independent real variables whose graph is a *surface* in $\mathbb{C}^{n+1} \cong \mathbb{R}^{2n+2}$, with the tangent plane spanned by two real vectors $\operatorname{Re} F, \operatorname{Im} F$. To derive this theorem from the standard results on existence, uniqueness and differentiability of solutions of *smooth* ordinary differential equations in the *real domain*, one should use rather deep results on *integrability of distributions*; see Remark 2.10 below. Rather unexpectedly, the direct proof is *simpler* than in the real case in the part concerning dependence on initial conditions. This proof is given in the next subsection. The main idea of this proof, as well as many other proofs below, is the *contracting map principle.*

1B. Contracting map principle. Consider the linear space $\mathcal{A}(D_\rho)$ of functions holomorphic in the polydisk D_ρ and continuous on its closure,

$$\mathcal{A}(D_\rho) = \{f\colon D_\rho \to \mathbb{C} \text{ holomorphic in } D_\rho \text{ and continuous on } \overline{D_\rho}\}. \tag{1.3}$$

This space is naturally equipped with the supremum-norm,

$$\|f\|_\rho = \max_{z \in D_\rho} |f(z)|, \qquad z = (z_1, \dots, z_n) \in \mathbb{C}^n, \tag{1.4}$$

and thus naturally a subspace of the *complete normed* (i.e., Banach) space $C(\overline{D_\rho})$ of continuous complex-valued functions. Though holomorphic functions may have very complicated boundary behavior and thus $\mathcal{A}(U) \subsetneqq \mathcal{O}(U)$, they are continuous and therefore for any smaller domain U' relatively compact in U (i.e., when $\overline{U'} \Subset U$), there is an obvious inclusion $\mathcal{A}(U') \supset \mathcal{O}(U)$.

Theorem 1.2. *The space $\mathcal{A}(D_\rho)$ and its vector counterparts $\mathcal{A}^m(D_\rho) = \bigoplus_{m \text{ times}} \mathcal{A}(D_\rho)$ are complete (Banach) spaces.*

Proof. Any fundamental sequence in $\mathcal{A}(D_\rho)$ is by definition fundamental in the Banach space $C(\overline{D_\rho})$ and has a uniform limit in the latter space. By the Weierstrass compactness principle [**Sha92**], this limit is again holomorphic in D_ρ, i.e., belongs to $\mathcal{A}(D_\rho)$. □

A map F of a metric space $\mathcal{M}$ into itself is called *contracting*, if for some positive real number $\lambda < 1$ and all $u, v \in \mathcal{M}$ the inequality $\operatorname{dist}(F(u), F(v)) \leqslant \lambda \operatorname{dist}(u, v)$ holds. A point $w \in \mathcal{M}$ is *fixed* (by F), if $F(w) = w$.

Theorem 1.3 (Contracting map principle). *Any contracting map $F\colon M \to M$ of a complete metric space M has a unique fixed point in $\mathcal{M}$.*

This fixed point is the limit of any sequence of iterations $u_{k+1} = F(u_k)$, $k = 0, 1, 2, \dots$ beginning with an arbitrary initial point $u_0 \in M$.

Proof. For any initial point $u_0 \in M$, the sequence u_k, $k = 1, 2, \dots$ is fundamental, since $\operatorname{dist}(u_k, u_{k+1}) \leqslant \lambda^k \operatorname{dist}(u_0, u_1)$ and by the triangle inequality $\operatorname{dist}(u_k, u_l) \leqslant \operatorname{dist}(u_0, u_1)\lambda^k/(1-\lambda)$ for any $k < l$. By completeness assumption, the sequence u_k converges to a limit $w \in M$. Since F is continuous, passing to the limit in the identity $u_{k+1} = F(u_k)$ yields $w = F(w)$. If w_1, w_2 are two fixed points, then $\operatorname{dist}(w_1, w_2) \leqslant \lambda \operatorname{dist}(F(w_1), F(w_2)) = \lambda \operatorname{dist}(w_1, w_2)$ which is possible only if $\operatorname{dist}(w_1, w_2) = 0$, i.e., when $w_1 = w_2$. □

1C. Picard operators and their contractivity. The exposition below is based on [**Arn78**, §31] with minor modifications.

Consider the equation (1.1) defined in a domain U. Denote by $D_\varepsilon = \{|z - x_0| < \varepsilon,\ |t - t_0| < \varepsilon\} \subset \mathbb{C}^{n+1}$ a polydisk centered at the point $(t_0, x_0) \in U$ and small enough to belong to U.

Definition 1.4. The *Picard operator* $\mathbf{P}$ associated with the differential equation (1.1) and the initial value $(t_0, z_0) \in U$, is the operator $f \mapsto \mathbf{P}f$ defined by the integral formula

$$(\mathbf{P}f)(s, z) = z + \int_{t_0}^{s} F(t, f(t, z))\, dt \tag{1.5}$$

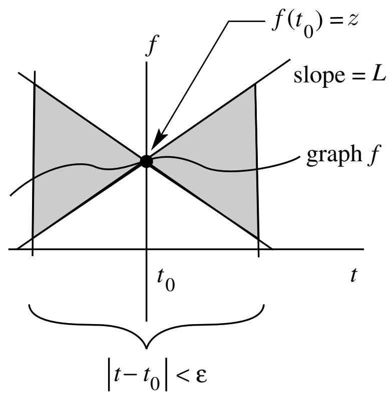

Figure I.1. Domain of definition of Picard iterations (in the intersection with the hyperplane $z = \text{const}$)

for all vector functions $f(t, z)$ the expression in the right hand side makes sense.

We will now construct a complete metric space invariant by $\mathbf{P}$, on which this operator is contracting. Denote by L_0 and L_1 the bounds for the magnitude of F and its Lipschitz constant in U: for any $(t, x), (t, x') \in U$,

$$|F(t,z)| \leqslant L_0, \qquad |F(t,z) - F(t,z')| \leqslant L_1\,|z - z'|. \tag{1.6}$$

Denote by $\mathcal{M}$ the subspace of the space $\mathcal{A}^n(D_\varepsilon)$ which consists of the functions satisfying the additional inequality

$$|f(t,z) - z| \leqslant L_0\,|t - t_0|. \tag{1.7}$$

This space is complete in the metric induced by the norm $\|\cdot\|_\varepsilon$ inherited from the ambient space $\mathcal{A}^n(D_\varepsilon)$ (Exercise 1.3).

Lemma 1.5. *If the polydisk D_ε is sufficiently small, the Picard operator $\mathbf{P}$ given by the integral* (1.5)*, is well defined and contracting on $\mathcal{M}$.*

More precisely, for sufficiently small ε its contraction factor λ does not exceed εL_1, where L_1 is the Lipschitz constant for F in U.

Proof. Explicit majorizing of the integral shows that

$$|\mathbf{P}f(s,z) - z| \leqslant L_0 \int_{t_0}^{s} |dt| \leqslant L_0\,|s - t_0| \leqslant L_0\varepsilon,$$

so if ε is chosen sufficiently small, the operator $\mathbf{P}$ is well defined on $\mathcal{M}$ and maps this space into itself. For any two vector functions f, f' defined on such a small polydisk D_ε, we have by virtue of the same estimate

$$\|\mathbf{P}f - \mathbf{P}f'\| = \sup_{|s-t_0|<\varepsilon} \int_{t_0}^{s} L_1\,|f(t,z) - f'(t,z)|\,|dt| \leqslant \varepsilon L_1\,\|f - f'\|.$$

If $\varepsilon L_1 < 1$, the operator $\mathbf{P}$ is contracting. □

Proof of Theorem 1.1. Assume ε is so small that the $\varepsilon L_1 < 1$ so that by Lemma 1.5, the Picard operator $\mathbf{P}$ is contracting. By Theorem 1.2 the fixed point of this operator (which exists by Theorem 1.3 and Lemma 1.5) is a *holomorphic* vector function $f\colon D_\varepsilon \to \mathbb{C}^n$ that satisfies the integral equation

$$f(s,z) = z + \int_{t_0}^{s} F(t, f(t,z))\,dt, \qquad |s-t_0| < \varepsilon, \quad |z - x_0| < \varepsilon. \tag{1.8}$$

For each fixed z, the function $\varphi_z(t) = f(t,z)$ clearly satisfies both the initial condition (1.2) with $x_0 = z$ and the differential equation (1.1). By construction, it depends holomorphically on the initial condition z.

To prove holomorphic dependence on additional parameters, one can treat them as fictitious dependent variables. Assume that the vector function $F = F(t,x,y)$ depends holomorphically on additional parameters $y \in \mathbb{C}^m$, and consider the initial value problem (recall that the dot means the derivative $\frac{d}{dt}$)

$$\begin{cases} \dot x = F(t,x,y), & x(t_0) = x_0, \\ \dot y = 0, & y(t_0) = y_0. \end{cases} \tag{1.9}$$

The solution of this initial value problem is a function $f(t,x,y,x_0,y_0)$ holomorphically depending on all variables. □

Remark 1.6. For a differential equation with the right hand side $F(t,x)$ the *shifted solution* $x'(t) = x(t-y)$, $y \in \mathbb{C}^1$, satisfies the shifted equation $\dot x' = F(t-y, x')$ which analytically depends on the parameter y. By Theorem 1.1, this shows that solutions of the initial value problem depend holomorphically also on the t-component of the initial point $(t_0, x_0) \in U$.

1D. Principal example: exponential formula for linear systems. The proof of the existence theorem is *constructive*: the solution of a differential equation is obtained as the uniform limit of its *Picard approximations*, iterations of the Picard operator.

In the simplest case of a differential equation with *constant* (i.e., independent of t,x,y) right hand side $F = \text{const} \in \mathbb{C}^n$ the Picard approximations stabilize immediately: if $f_0(t,v) = v$, then $f_1(t,v) = f_2(t,v) = \cdots = v + (t-t_0)F$.

A *linear system with constant coefficients* is the system of equations

$$\dot x = Ax, \qquad x \in \mathbb{C}^n, \quad A \in \operatorname{Mat}(n, \mathbb{C}) \tag{1.10}$$

where $A = \|a_{ij}\|$ is a constant (independent of t and x) $(n \times n)$-matrix with complex entries. Reasoning by induction, one can see that the Picard approximations for the solution of (1.10) which start with the constant initial term $f_0(t,v)=v$, have the form

$$f_k(t,v)=\Bigl(E+tA+\tfrac{t^2}{2!}A^2+\cdots+\tfrac{t^k}{k!}A^k\Bigr)\,v. \tag{1.11}$$

Indeed,

$$\begin{aligned}\mathbf{P}f_k(t,v)&=v+\int_0^t A\cdot\bigl(E+sA+\cdots+\tfrac{s^k}{k!}A^k\bigr)v\,ds\\ &=Ev+\bigl(tA+\cdots+\tfrac{t^{k+1}}{(k+1)!}A^{k+1}\bigr)\,v=f_{k+1}(t,v).\end{aligned}$$

These formulas motivate the following fundamental object.

Definition 1.7 (matrix exponential)**.** For an arbitrary constant matrix $A\in\operatorname{Mat}(n,\mathbb{C})$ its *exponential* $\exp A$ is the sum of the infinite (matrix) series

$$\exp A=E+A+\frac{1}{2!}A^2+\cdots+\frac{1}{k!}A^k+\cdots. \tag{1.12}$$

Since $|A^k|\leqslant|A|^k$ and since the factorial series $\sum_{k\geqslant0}r^k/k!$ converges absolutely for all values $r\in\mathbb{R}$, the matrix series (1.12) converges absolutely on the complex linear space $\operatorname{Mat}(n,\mathbb{C})\cong\mathbb{C}^{n^2}$ for any finite n.

Note that for any two *commuting* matrices A,B their exponents satisfy the *group identity*

$$\exp(A+B)=\exp A\cdot\exp B=\exp B\cdot\exp A. \tag{1.13}$$

This can be proved by substituting A,B instead of two scalars a,b into the formal identity obtained by expansion of the law $e^ae^b=e^{a+b}$.

The explicit formula (1.11) for Picard approximations for the linear system (1.10) immediately proves the following theorem.

Theorem 1.8. *The solution of the linear system $\dot x=Ax$, $A\in\operatorname{Mat}(n,\mathbb{C})$, with the initial value $x(0)=v$ is given by the matrix exponential,*

$$x(t)=(\exp tA)\,v,\qquad t\in\mathbb{C},\quad v\in\mathbb{C}^n.\quad\Box \tag{1.14}$$

Remark 1.9. Computation of the matrix exponential can be reduced to computation of a matrix polynomial of degree $\leqslant n-1$ and exponentials of eigenvalues of A. Indeed, assume that A has a Jordan normal form $A=\varLambda+N$, where $\varLambda=\operatorname{diag}\{\lambda_1,\dots,\lambda_n\}$ is the diagonal part and N the upper-triangular (nilpotent) part *commuting* with $\varLambda$. Then $\exp\varLambda$ is a diagonal matrix with the exponentials of the eigenvalues of $\varLambda$ on the diagonal, $N^n=0$

by nilpotency, and therefore

$$\exp[t(\Lambda+N)] = \exp t\Lambda \cdot \exp tN$$
$$= \begin{pmatrix} \exp t\lambda_1 & & \\ & \ddots & \\ & & \exp t\lambda_n \end{pmatrix} \cdot \left(E + tN + \frac{t^2}{2!}N^2 + \cdots + \frac{t^{n-1}}{(n-1)!}N^{n-1}\right). \tag{1.15}$$

This provides a practical way of solving linear systems with constant coefficients: components of any solution in any basis are linear combinations of *quasipolynomials* $t^k \exp t\lambda_j$, $0 \leqslant k \leqslant n-1$ with complex coefficients.

Remark 1.10 (Liouville–Ostrogradskii formula)**.** By direct inspection of the formula (1.15) we conclude that

$$\forall A \in \operatorname{Mat}(n,\mathbb{C}) \qquad \det\exp A = \exp\operatorname{tr} A. \tag{1.16}$$

Indeed, $\det\exp A = \det\exp\Lambda \cdot \det\exp N = \prod_{i=1}^n \exp\lambda_i \cdot 1 = \exp\operatorname{tr}\Lambda = \exp\operatorname{tr} A$, since the matrix polynomial $\exp N$ is upper triangular with units on the diagonal.

1E. Flow box theorem. Let $f(t,x_0)$ be the holomorphic vector function solving the initial value problem (1.2) for the differential equation (1.1).

Definition 1.11. The *flow map* for a differential equation (1.1) is the vector function of $n+2$ complex variables (t_0,t_1,v) defined when $(t_0,x) \in U$ and $|t_0-t_1|$ is sufficiently small, by the formula

$$(t_0,t_1,v) \mapsto \Phi_{t_0}^{t_1}(v) = f(t_1,v), \tag{1.17}$$

where $f(t,v)$ is the fixed point of the Picard operator $\mathbf{P}$ as in (1.8) associated with the initial point t_0.

In other words, $\Phi_{t_0}^{t_1}(v)$ is the value $\varphi(t)$ which takes the solution of the initial value problem with the initial condition $\varphi(t_0) = v$, at the point t_1 sufficiently close to t_0.

Example 1.12. For a linear system (1.10) with constant coefficients, the flow map is linear:

$$\Phi_{t_0}^{t_1}(v) = [\exp(t_1-t_0)A]\,v.$$

This map is defined for *all* values of t_0, t_1, v.

By Theorem 1.1, Φ is a holomorphic map. Since the solution of the initial value problem is unique, it obviously must satisfy the functional equation

$$\Phi_{t_1}^{t_2}(\Phi_{t_0}^{t_1}(x)) = \Phi_{t_0}^{t_2}(x) \tag{1.18}$$

for all t_1, t_2 sufficiently close to t_0 and all x sufficiently close to x_0. Since for any x the vector function $t \mapsto \varphi_x(t) = \Phi_{t_0}^t(x)$ is a solution of (1.1), we have

$$\frac{\partial}{\partial t}\bigg|_{t=t_0,\, x=x_0} \Phi_{t_0}^t(x) = -\frac{\partial}{\partial t_0}\bigg|_{t=t_0,\, x=x_0} \Phi_{t_0}^t(x) = F(t_0, x_0).$$

From the integral equation (1.8) it follows that

$$\Phi_{t_0}^t(x_0) = x_0 + (t - t_0)F(t_0, x_0) + o(|t - t_0|), \tag{1.19}$$

and therefore the Jacobian matrix of Φ with respect to the x-variable is

$$\left(\frac{\partial \Phi_{t_0}^t(x)}{\partial x}\right)_{t=t_0,\, x=x_0} = E. \tag{1.20}$$

Differential equations can be transformed to each other by various transformations. The most important is the (bi)holomorphic equivalence, or holomorphic conjugacy.

Definition 1.13. Two differential equations, (1.1) and another such equation

$$\dot{x}' = F'(t', x'), \qquad (t', x') \in U', \tag{1.21}$$

are *conjugated* by the biholomorphism $H\colon U \to U'$ (the *conjugacy*), if H sends any integral trajectory of (1.1) into an integral trajectory of (1.21).

Two systems are *holomorphically equivalent* in their respective domains, if there exists a biholomorphic conjugacy between them.

Clearly, biholomorphically conjugate systems are indistinguishable in everything that concerns properties invariant by biholomorphisms. Finding a simple system biholomorphically equivalent to a given one, is therefore of paramount importance.

Theorem 1.14 (Flow box theorem)**.** *Any holomorphic differential equation* (1.1) *in a sufficiently small neighborhood of any point is biholomorphically conjugated by a suitable biholomorphic conjugacy* $H\colon (t, x) \mapsto (t, h(t, x))$ *preserving the independent variable* t*, to the trivial equation*

$$\dot{x}' = 0. \tag{1.22}$$

Proof of the theorem. Consider the map $H'\colon \mathbb{C}^{n+1} \to \mathbb{C}^{n+1}$ which is defined near the point (t_0, x_0) using the flow map (1.17) for the equation (1.1),

$$H'\colon (t, x') \mapsto (t, \Phi_{t_0}^t(x')), \qquad (t, x') \in (\mathbb{C}^{n+1}, (t_0, x_0)).$$

By construction, it takes the lines $x' = \text{const}$ parallel to the t-axis, into integral trajectories of the equation (1.1), while preserving the value of t.

The Jacobian matrix $\partial H'(t, x')/\partial(t, x')$ of the map H' at the point (t_0, x_0) has by (1.20) the form $\left(\begin{smallmatrix} 1 & \\ * & E \end{smallmatrix}\right)$ and is therefore invertible.

Thus H' restricted on a sufficiently small neighborhood of the point (t_0, x_0), is a biholomorphic conjugacy between the trivial system (1.21), whose solutions are exactly the lines $x' = \text{const}$, and the given system (1.1). The inverse map also preserves t and conjugates (1.1) with (1.21). □

1F. Vector fields and their equivalence. The above constructions after small modification become more transparent in the *autonomous* case, when the vector function $x \mapsto F(x)$ which is now independent of t, can be considered as a *holomorphic vector field* on its domain $U \subseteq \mathbb{C}^n$. The space of vector fields holomorphic in a domain $U \subseteq \mathbb{C}^n$ will be denoted by $\mathcal{D}(U)$, while the notation $\mathcal{D}(\mathbb{C}^n, x_0)$ is reserved for the space of germs of holomorphic vector fields at a specific point $x_0 \in \mathbb{C}^n$, usually the origin, $x_0 = 0$.

In the autonomous case, translation of the independent variable preserves solutions of the equation

$$\dot{x} = F(x), \qquad F\colon U \to \mathbb{C}^n, \tag{1.23}$$

so the flow map $\varPhi_{t_0}^{t_1}$ actually depends only on the difference $t = t_1 - t_0$ and hence will be denoted simply by $\varPhi^t(\cdot) = \varPhi_0^t(\cdot)$. The functional identity (1.18) takes the form

$$\varPhi^t(\varPhi^s(x)) = \varPhi^{t+s}(x), \qquad t, s \in (\mathbb{C}, 0),\ x \in (\mathbb{C}^n, x_0), \tag{1.24}$$

which means that the maps $\{\varPhi^t\}$ form a one-parametric *pseudogroup* of biholomorphisms. ("Pseudo" means that the composition in (1.24) is not always defined. The pseudogroup is a true group, $\varPhi^t \circ \varPhi^s = \varPhi^{t+s}$, if the maps $\varPhi^t$ are globally defined for all $t \in \mathbb{C}$. For more details on pseudogroups see §6**D**).

For autonomous equations it is natural to consider biholomorphisms that are *time-independent*.

Definition 1.15. Two holomorphic vector fields, $F \in \mathcal{D}(U)$ and $F' \in \mathcal{D}(U')$ defined in two domains $U, U' \subseteq \mathbb{C}^n$, are *biholomorphically equivalent* if there exists a biholomorphic map $H\colon U \to U'$ conjugating their respective flows,

$$H \circ \varPhi^t = \varPhi'^t \circ H \tag{1.25}$$

whenever both sides are defined. The biholomorphism H is said to be a *conjugacy* between F and F'.

A conjugacy H maps *phase* curves of the first field into phase curves of the second field; in a similar way the suspension

$$\operatorname{id} \times H\colon (\mathbb{C}, 0) \times U \to (\mathbb{C}, 0) \times U', \qquad (t, x) \mapsto (t, H(x)),$$

maps *integral* curves of the two fields into each other. Differentiating the identity (1.25) in t at $t = 0$, we conclude that the differential $dH(x)$ of a holomorphic conjugacy sends the vector $v = F(x)$ of the first field, attached to a point $x \in U$, to the vector $v' = F'(x')$ of the second field *at the appropriate point* $x' = H(x)$. In the coordinates this property takes the form of the identity

$$H_*(x) \cdot F(x) = F'(H(x)), \qquad H_*(x) = \left(\tfrac{\partial H}{\partial x}\right) = \left(\tfrac{\partial h_i}{\partial x_j}\right), \tag{1.26}$$

in which the Jacobian matrix $H_*(x) = \left(\frac{\partial H}{\partial x}\right)$ is involved. The formula (1.26) is sometimes used as the alternative *definition* of the holomorphic equivalence. The third (algebraic, in some sense most natural) way to introduce this equivalence is explained in the next section.

1G. Vector fields as derivations. It is sometimes convenient to define vector fields in a way independent of the coordinates. Each vector field $F = (F_1, \dots, F_n)$ in a domain $U \subset \mathbb{C}^n$ defines a *derivation* $\mathbf{F} \in \operatorname{Der} \mathcal{O}(U)$ of the $\mathbb{C}$-algebra $\mathcal{O}(U)$ of functions holomorphic in U, by the formula

$$\mathbf{F} f(x) = \sum_{j=1}^{n} F_j(x) \, \frac{\partial f}{\partial x_j}. \tag{1.27}$$

We often identify the holomorphic vector field F with the components F_i with the corresponding differential operator $\mathbf{F} = \sum F_j \frac{\partial}{\partial x_j}$.

Derivations can be defined in purely algebraic terms as $\mathbb{C}$-linear maps of the algebra $\mathcal{O}(U)$ satisfying the Leibnitz identity,

$$\mathbf{F}(fg) = f(\mathbf{F}g) + (\mathbf{F}f)g.$$

Indeed, any $\mathbb{C}$-linear operator with this property in any coordinate system $(x_1, \dots, x_n)$ defines n functions $F_j = \mathbf{F}x_j$ and (obviously) sends all constants to zero. Any analytic function f can be written $f(x) = f(a) + \sum_1^n h_j(x)\,(x_j - a_j)$ with $h_j(a) = \frac{\partial f}{\partial x_j}(a)$. Applying the Leibnitz rule, we conclude that $(\mathbf{F}f)(a) = \sum_j F_j h_j(a) + 0 \cdot \mathbf{F}h_j = \sum_j F_j \frac{\partial f}{\partial x_j}(a)$, as claimed.

A similar algebraic description can be given for holomorphic maps. With any holomorphic map $H \colon U \to U'$ between two domains $U, U' \subseteq \mathbb{C}^n$ one can associate the *pullback operator* $\mathbf{H} \colon \mathcal{O}(U') \to \mathcal{O}(U)$, acting on $f' \in \mathcal{O}(U')$ by composition, $(\mathbf{H}f')(x) = f'(H(x))$. This operator is a *homomorphism* of commutative $\mathbb{C}$-algebras, a $\mathbb{C}$-linear map respecting multiplication (i.e., $\mathbf{H}(f'g') = \mathbf{H}f' \cdot \mathbf{H}g'$ for any $f', g' \in \mathcal{O}(U')$). Conversely, any continuous homomorphism $\mathbf{H}$ between the two algebras is induced by a holomorphic map $H = (h_1, \dots, h_n)$ with $h_i = \mathbf{H}x_i$, where $x_i \in \mathcal{O}(U')$ are the coordinate functions (restricted on U'). The map H is a biholomorphism if and only if $\mathbf{H}$ is an isomorphism of $\mathbb{C}$-algebras.

In this language the action of biholomorphisms on vector fields can be described as a simple *conjugacy of operators*: two derivations $\mathbf{F}$ and $\mathbf{F}'$ of the algebras $\mathcal{O}(U)$ and $\mathcal{O}(U')$ respectively, are said to be conjugated by the biholomorphism $H\colon U \to U'$, if

$$\mathbf{F} \circ \mathbf{H} = \mathbf{H} \circ \mathbf{F}' \tag{1.28}$$

as two $\mathbb{C}$-linear operators from $\mathcal{O}(U')$ to $\mathcal{O}(U)$.

Another advantage of this invariant description is the possibility of defining the *commutator* of two vector fields naturally, as the commutator of the respective differential operators. One can immediately verify that $[\mathbf{F}, \mathbf{F}'] = \mathbf{F}\mathbf{F}' - \mathbf{F}'\mathbf{F}$ satisfies the Leibnitz identity as soon as $\mathbf{F}, \mathbf{F}'$ do, and hence corresponds to a vector field. In coordinates the commutator takes the form

$$[F, F'] = \left(\frac{\partial F'}{\partial x}\right) F - \left(\frac{\partial F}{\partial x}\right) F'. \tag{1.29}$$

Example 1.16. For any two $\mathbf{F} = Ax$, $\mathbf{F}' = A'x$ *linear* vector fields, their commutator $[\mathbf{F}, \mathbf{F}']$ is again a linear vector field with the linearization matrix $A'A - AA'$. It coincides (modulo the sign) with the usual matrix commutator $[A, A']$.

1H. Rectification of vector fields. A straightforward counterpart of the Flow box Theorem 1.14 for holomorphic vector fields holds only if the field is nonvanishing.

Definition 1.17. A point x is a *singular point* (*singularity*) of a holomorphic vector field F, if $F(x_0) = 0$. Otherwise the point is *nonsingular.*

Theorem 1.18 (Rectification theorem)**.** *A holomorphic vector field F is holomorphically equivalent to the constant vector field $F'(x') = (1, 0, \dots, 0)$ in a sufficiently small neighborhood of any nonsingular point.*

Proof. The flow Φ' of the constant vector field F' can be immediately computed: $(\Phi')^t(x') = x' + t \cdot (1, 0, \dots, 0)$. Consider any affine hyperplane $\Pi \subset U$ passing through x_0 and transversal to $F(x_0)$ and the hyperplane $\Pi' = \{x'_1 = 0\}$. Let $t = x'_1 \colon \mathbb{C}^n \to \mathbb{C}$ be the function equal to the first coordinate in $\mathbb{C}^n$, so that $(\Phi')^{-t}(x') \in \Pi'$. Let $h'\colon \Pi' \to \Pi$ be *any* biholomorphism (e.g., linear invertible map). Then the map

$$H' = \Phi^t \circ h \circ (\Phi')^{-t}, \qquad t = t(x'),$$

is a holomorphic map that sends any (parameterized) trajectory of F', passing through a point $x' \in \Pi'$, to the parameterized trajectory of F passing through $x = h(x')$. Being composition of holomorphic maps, H' is also holomorphic, and coincides with h' when restricted on Π'. It remains to notice that the differential $dH'(x_0)$ maps the vector $(1, 0, \dots, 0)$ transversal to Π', to the vector $F(x_0)$ transversal to Π. This observation proves that H' is invertible in some sufficiently small neighborhood U of x_0, and the inverse map H conjugates F in U with F' in $H(U)$. □

1I. One-parametric groups of holomorphisms. The Rectification theorem from §1 can be formulated in the language of germs as follows: *Two germs of holomorphic vector fields at nonsingular points are always holomorphically equivalent to each other.* In particular, any germ of a holomorphic vector field at a nonsingular point is holomorphically equivalent to the germ of a nonzero constant vector field.

Because of this "triviality" of local description of nonsingular vector fields, we will mostly be interested in germs of vector fields at the *singular points*. The first result is existence of germs of the flow maps $\varPhi^t$ at the singular point, for all values of $t \in \mathbb{C}$.

Denote by $\operatorname{Diff}(\mathbb{C}^n, 0)$ the group of germs of holomorphic self-maps $H\colon (\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$ equipped with the operation of composition (which is always defined).

Proposition 1.19. *If $F \in \mathcal{D}(\mathbb{C}^n, 0)$ is the germ of a holomorphic vector field which is singular (i.e., $F(0) = 0$), then the germs of the flow maps $\varPhi^t(\cdot)$ are defined for all $t \in \mathbb{C}$ and form a one-parametric subgroup of the group $\operatorname{Diff}(\mathbb{C}^n, 0)$ of germs of biholomorphic self-maps: $\varPhi^t \circ \varPhi^s = \varPhi^{t+s}$ for any $t, s \in \mathbb{C}$.*

Proof. The existence of the flow maps $\varPhi^t$ for all sufficiently small $t \in (\mathbb{C}, 0)$, the possibility of their composition, and validity of the group identity for such small t all follow from Theorem 1.1 and the fact that $\varPhi^t(x_0) = x_0$.

For an arbitrary large value of $t \in \mathbb{C}$ we may define $\varPhi^t$ as the composition of germs of the flow maps $\varPhi^{t_i}$, $i = 1, \dots, N$, taken in any order, where the complex numbers t_i are sufficiently small to satisfy conditions of Theorem 1.1 but added together give t. From the local group identity it follows that the definition does not depend on the particular choice of t_i and preserves the group property. □

Remark 1.20. Every germ of a self-map $H \in \operatorname{Diff}(\mathbb{C}^n, 0)$ uniquely defines an *auto*morphism $\mathbf{H} \in \operatorname{Aut} \mathcal{O}(\mathbb{C}^n, 0)$ of the commutative algebra of holomorphic germs acting by substitution, $\mathbf{H}f = f \circ H$.

Proposition 1.19 translates into the algebraic language as follows: for any derivation $\mathbf{F} \in \operatorname{Der} \mathcal{O}(\mathbb{C}^n, 0)$ of the algebra of holomorphic germs there exist a one-parametric subgroup $\{\mathbf{H}^t\colon t \in \mathbb{C}\} \subset \operatorname{Aut} \mathcal{O}(\mathbb{C}^n, 0)$ of automorphisms of this algebra, such that $\frac{d}{dt}\big|_{t=0} \mathbf{H}^t = \mathbf{F}$.

For the reasons to be explained below in §3**C**, the subgroup of automorphisms $\mathbf{H}^t$ is often referred to as the *exponent*, $\mathbf{H}^t = \exp(t\mathbf{F})$, of the derivation **F**. Respectively, the flow (germs of self-maps) will be sometimes denoted by the exponent, $\Phi^t = \exp(tF)$, of the corresponding vector field F.

Exercises and Problems for §1.

Exercise 1.1. Let $a \in U$ be a nonsingular point of a holomorphic vector field $F \in \mathcal{D}(U)$. A trajectory of the vector field is the projection of the graph of the solution into the domain of the field along the time axis.

Prove that the trajectory passing through a is either the line $x = a$, or can be represented as the graph of a function $y = \varphi_a(x)$ having an *algebraic* ramification point of some finite order ν. Express ν in terms of orders of the components of the field F at a.

Exercise 1.2. Let $P\colon (\mathbb{C}^n, 0) \to (\mathbb{C}^{n-1}, 0)$ be a holomorphic epimorphism (i.e., map of rank $n-1$) constant along trajectories of an analytic vector field $F \in \mathcal{D}(\mathbb{C}^n, 0)$. Construct explicitly the rectifying chart for F.

Exercise 1.3. Prove that the space $\mathcal{M}$ of functions satisfying the inequality (1.7), is indeed complete.

Exercise 1.4. Two linear vector fields in $\mathbb{C}^n$ are holomorphically equivalent in some domains containing the origin. Prove that these fields are *linear* equivalent, i.e., that there exists a linear map $H \in \mathrm{GL}(n, \mathbb{C})$ conjugating them.

Exercise 1.5. Prove that if two germs of vector fields at a singular point are analytically equivalent, then the eigenvalues of these fields at the singular point coincide.

Exercise 1.6. Prove that the vector field $F(z) = z^2 \frac{\partial}{\partial z}$ is holomorphic on the Riemann sphere $\mathbb{P}^1 = \mathbb{C} \cup \{\infty\}$. Compute the flow of this field.

Problem 1.7. Give a complete analytic classification of the holomorphic flows on the Riemann sphere $\mathbb{P}^1$ (i.e., construct a list, finite or infinite, of flows such that every holomorphic flow in analytically equivalent to one of the flows from the list, while any two different flows in the list are *not* holomorphically equivalent.

Exercise 1.8. Prove that the constant holomorphic vector fields $\frac{\partial}{\partial z}$ on the two tori $\mathbb{T}_1 = \mathbb{C}/(\mathbb{Z} + i\mathbb{Z})$ and $\mathbb{T}_2 = \mathbb{C}/(\mathbb{Z} + 2i\mathbb{Z})$, are not holomorphically equivalent.

2. Holomorphic foliations and their singularities

By the Existence/Uniqueness Theorem 1.1, any open connected domain $U \subseteq \mathbb{C}^n$ with a holomorphic vector field F defined on it, can be represented as the disjoint union of connected phase curves passing through all points of U. The Rectification Theorem 1.18 provides a local model for the geometric object called *foliated space* of simply *foliation*. A systematic treatment of foliations can be found, for instance, in [**Tam92, CC03**].