$\mathcal{F}$, associated with the loop $\gamma \subset L$, by construction coincides with the map f which is transformed into the self-map. □

The construction can be modified by a number of ways, while keeping the principal idea the same. If $\widetilde{M}$ is a manifold with a foliation $\mathcal{F}_0$ on it, and $\mathbf{f}\colon M_0 \to M_1$ is a biholomorphic map between open subsets of $\widetilde{M}$, which is an automorphism of the foliation $\mathcal{F}_0$, then the quotient space $M = \widetilde{M}/\mathbf{f}$ is a new manifold with a different topology, which carries a holomorphic foliation on it.

Exercises and Problems for §2.

Exercise 2.1. Let $S \subset (\mathbb{C}^n, 0)$ be the germ of an irreducible analytic curve and γ an injective analytic parametrization. Prove that any other holomorphic map $\gamma'\colon (\mathbb{C}^1, 0) \to (\mathbb{C}^n, 0)$ with the range in S differs from γ by a holomorphic map $h\colon (\mathbb{C}^1, 0) \to (\mathbb{C}^1, 0)$ so that $\gamma' = \gamma \circ h$.

Problems 2.2–2.7 together constitute a proof of the Frobenius Theorem 2.9.

Problem 2.2. Prove that vector fields generating an integrable distribution, are *in involution*, i.e., always satisfying condition (2.4).

Prove that Pfaffian forms generating an integrable distribution, are in involution, i.e., satisfy the conditions (2.5).

Problem 2.3. Prove that two holomorphic vector fields $F, F' \in \mathcal{D}(M)$ on a holomorphic manifold M, have identically zero commutator, $[F, F'] \equiv 0$, if and only if their flows $\exp(tF), \exp(t'F') \in \operatorname{Diff}(M)$ commute for all complex values of $t, t' \in \mathbb{C}$.

Formulate and prove an analog of this result for *incomplete* vector fields (i.e., when the flows are not globally defined for all values of t, t', as in the case where $U \subseteq \mathbb{C}^2$ is a noninvariant planar domain).

Problem 2.4. Prove that any tuple of everywhere linearly independent commuting vector fields generates an integrable distribution tangent to leaves of a holomorphic foliation.

Problem 2.5. Let $F_1, \dots, F_k$ be holomorphic everywhere linearly independent vector fields in involution (i.e., satisfying condition (2.4)).

Construct another tuple of holomorphic vector fields $F_1', \dots, F_k'$ spanning the same distribution, such that the fields $[F_i', F_j'] \equiv 0$ for all $1 \leqslant i, j \leqslant k$.

Prove that vector fields in involution generate an integrable distribution.

Problem 2.6. Prove that for any differential 1-form ω and two vector fields F, G on a manifold M,

$$d\omega(F, G) = F\,\omega(G) - G\,\omega(F) - \omega([F, G]) \tag{2.11}$$

(the right hand side contains the evaluation of ω on the fields F, G and $[F, G]$ and their derivatives along G and F).

Problem 2.7. Prove that a tuple of everywhere linearly independent 1-forms satisfying (2.5), defines an integrable distribution.

Exercise 2.8. Prove that a nonvanishing Pfaffian form ω in $\mathbb{C}^3$ defines an integrable distribution, if and only if $\omega \wedge d\omega = 0$.

Problem 2.9. Prove that each holonomy operator g corresponding to any separatrix of an integrable foliation $du = 0$ with an analytic potential $u \in \mathcal{O}(x, y)$, is periodic: some iterated power of g is identity.

Exercise 2.10. Construct two foliations having leaves with holomorphically conjugated holonomy groups, which are themselves not holomorphically conjugate in neighborhoods of the leaves.

Exercise 2.11. Is it always possible to rectify *simultaneously* two nonsingular vector fields? Two *commuting* nonsingular vector fields? Give a simple sufficient condition guaranteeing such simultaneous rectification.

Exercise 2.12. Consider the foliation $\{\omega = 0\}$ on $\mathbb{C}^2 = \{(z, t)\}$ defined by a *meromorphic* Pfaffian 1-form

$$\omega = \frac{dz}{z} - \sum_{j=0}^{n} \frac{\lambda_j \, dt}{t - a_j}, \qquad \lambda_j \in \mathbb{C}, \ \sum_{0}^{n} \lambda_j = 0,$$

and its extension on $\mathbb{C} \times \mathbb{P}^1$.

Prove that the projective line $L = \{0\} \times \mathbb{P}^1$ is a separatrix of this foliation carrying singular points $(0, a_j)$, $j = 0, \dots, n$. Compute the holonomy group of the leaf $L \smallsetminus$ (singular points).

Exercise 2.13. The same question about the foliation on $\mathbb{C}^m \times \mathbb{P}^1$ defined by the vector Pfaffian form

$$dz - \Omega z = 0, \qquad \Omega = \sum_{0}^{n} \frac{A_j \, dt}{t - a_j},$$

where $A_j \in \operatorname{Mat}(m, \mathbb{C})$ are *commuting* matrix residues of the meromorphic matrix 1-form Ω.

Problem 2.14. Consider the Riccati equation

$$\frac{dz}{dt} = a(t)\, z^2 + b(t)\, z + c(t), \qquad a, b, c \in \mathcal{M}(\mathbb{P}) \cong \mathbb{C}(t), \tag{2.12}$$

with meromorphic coefficients a, b, c having poles only on the finite point set $\Sigma \subseteq \mathbb{P}$. Is it true that solutions of this equation can be continued along any path on the t-plane, avoiding the singular locus Σ?

Prove that equation (2.12) defines a singular holomorphic foliation $\mathcal{F}$ on the compactified phase space $\mathbb{P}^1 \times \mathbb{P}^1$, which is transversal to any "vertical" projective line $\{t = a\}$, $a \notin \Sigma$. Show that each leaf of $\mathcal{F}$ can be continued over any path in the t-sphere, avoiding the singular locus. Prove that the induced transformation between any two cross-sections $\{t = a\} \times \mathbb{P}^1$ and $\{t = b\} \times \mathbb{P}^1$, $a, b \notin U$, is a well-defined Möbius transformation (fractional linear map $z \mapsto \frac{\alpha z + \beta}{\gamma z + \delta}$ with $\alpha\delta - \beta\gamma \neq 0$). Does $\mathcal{F}$ always possess a separatrix?

Exercise 2.15. How many separatrices a *homogeneous* vector field of degree r on $\mathbb{C}^2$ may have? How many separatrices a *generic* homogeneous vector field has?